Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Bernoulli Polynomials Method for Solving Integral Equations with Singular Kernel

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Received 28/09/2023, Revised 15/01/2024, Accepted 17/01/2024, Published Online First 20/05/2024

Abstract

There is always an interest in an effective technique to generate a numerical solution of integral equations with singular or weakly singular kernels more precisely because numerical methods have limitations. In this study, integral equations with singular or weakly singular kernels are solved using the Bernoulli polynomial approach. The primary goal of this study is to provide an approximate solution to such problems in the form of a polynomial in a series of straightforward steps. Also, assuming that the denominator of the kernel will never be zero or have an imaginary value due to the selected nodes of the unique two kernel variables. With the 4^{th} and 8^{th}-degree Bernoulli polynomials as an example, the current approach provides a solution very close to the exact solution in the test examples. While. The very modest magnitude of the errors in the test examples proves the effectiveness of the current strategy. Also, the ease with which a computer program can be implemented makes this technique very efficient. Another objective is to determine the efficiency of the proposed method by comparing it with various approaches. The approximated solution for integral equations with singular or weakly singular kernels is demonstrated to significantly converge to the precise ones by using the Bernoulli polynomial and is superior to those found by other stated approaches. This guarantees the originality and high accuracy of the suggested method. Also, the convergent of the proposed method is discussed. The programs are implemented using the MATLAB program R2018a.

Keywords: Abel's integral equation, Bernoulli polynomials, Integral equation, Singular kernel, Weakly singular kernel.

Introduction

Due to the wide range of applications of singular integral equations, they are used in many different areas of science. Examples of such fields where applied mathematics is used include the theory of elasticity, viscoelasticity, hydrodynamics, and other fields. Many singular integral equations have very difficult-to-use analytic solutions. Many research efforts have focused on developing approaches that are more effective and efficient for solving integral equations with singular and weakly singular kernels such as; To obtain an approximate solution to second-order integral equations with weakly singular kernels, power series with collocation were investigated [1]. Generalized weakly singular Fredholm integral equations of the second kind are solved using Euler-Maclaurin's summation formula [2]. Legendre multiwavelets are used to find the numerical solution of Cauchy-type singular integral equations of the 1st kind with generalized kernels [3]. The hybrid orthonormal Bernstein and block-pulse function wavelet methods were considered to solve the nonlinear Volterra integral equations (VIE) with weakly singular kernel [4]. For the linear VIE with a weakly singular convolution kernel, asymptotic expansions are derived using the Laplace transform [5]. Touchard polynomials were used to find an approximate solution for Linear VIE of the 1st and 2nd kind with a singular kernel [6]. A formula for second-order backward differentiation was proposed for the Volterra integro-differential equation with a weakly singular kernel solved by the sinc-collocation method [7]. Finally, a method of interpolation for weakly singular VIE of 2nd kind based on the barycentric Lagrange approach was proposed [8].

On the other side, several researchers solved various forms of integral equations using Bernoulli polynomials, such that; Orthonormal Bernoulli polynomials were utilized in some VIE applications to provide an approximate solution [9]. .Operational matrices of Bernoulli wavelet are used for solving linear stochastic Itô-VIE [10]. Bernoulli wavelet based on the numerical method was developed for the solution of Abel's integral equations [11]. A numerical solution of weakly singular fractional partial integro-differential equations using Bernoulli polynomials was established [12]. Bernoulli polynomials and Bernoulli numbers are used to construct the orthonormal polynomials by using Gram–Schmidt orthogonalization to solve Abel-type integral equations [13]. For the solution of variable-order fractional optimal control affine problems, Bernoulli polynomials were used [14]. A method for solving the nonlinear Volterra-Fredholm-Hammerstein integral equations using orthonormal Bernoulli polynomials is considered [15]. To achieve the approximate solution of the linear singular stochastic Itô-VIE, the orthonormal Bernoulli collocation method was developed [16]. According to the Bernoulli polynomials, fractional-order Bernoulli wavelets are created and used to determine the numerical solution of the Caputo fractional order diffusion wave equations in their general form [17], The numerical approach for multi-term variable-order fractional differential equations that are based on Bernoulli polynomials are employed [18]. The first and second types of linear Abel integral equations were introduced and solved using Boubaker polynomials [19]. Finally, for solving the VIE of the second kind with a weakly singular kernel, a numerical technique combining a series solution and conformal mappings was presented [20] .

In this paper, Bernoulli polynomials are suggested to construct an approximate solution for the integral equation of singular and weakly singular kernel of the form:

$$u(x) = f(x) + \int_a^x K(x,t)u(t)dt \quad 1$$

or,

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt \quad 2$$

Such that $f(x)$ is a continuously defined function in the closed interval $[a,b]$, where $a \; and \; b$ are constant, and $u(x)$ is the unknown function defined on $L^2[a,b]$, to be determined, $K(x,t)$ is a singular or weakly singular kernel, not that the given kernel $k(x,t)$ takes the form $k(x,t) = \frac{1}{(x-t)^\alpha}$, $0 < \alpha < 1$ or $k(x,t) = \frac{t^{\mu-1}}{x^\mu}$, $0 < \mu < 1$. The following is how the paper is organized: Section 2 defines the Bernoulli polynomial as well as some frequently used formulas. A description of the suggested technique can be found in Section 3. Sections 4 and 5 include the algorithm and convergence analysis of the proposed method respectively. In Section 6, the suggested approach is used to solve a few test problems employing the new technique. A comparison with various approaches is also made. Finally, a result and recommendation are provided in section 7.

Materials and Methods

Bernoulli Polynomials:

Researchers have employed the Bernoulli polynomials to solve many different kinds of equations. Although these polynomials have certain useful properties, they miss the orthogonality property. In this part, an accurate description of Bernoulli polynomials is proposed and introduced:

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Over time, it has become clear that Bernoulli's numbers B_n and polynomials $B_n(z)$ are significant mathematical elements. It is commonly known that the generating function for the Bernoulli polynomials is[21]:

$$\frac{t}{e^{t}-1}e^{zt} = \sum_{k=0}^{\infty} B_k(z)\frac{t^k}{k!} \qquad 3$$

Note that when $z = 0, B_n = B_n(0)$ are called the Bernoulli numbers. Additionally, Bernoulli polynomials are described by [22]:

$$B_n(z) = \sum_{k=0}^{n} \binom{n}{k} B_k z^{n-k} \qquad 4$$

and the explicit formula:

$$B_n(z) = \sum_{i=0}^{n} \frac{1}{i+1} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (z+j)^n \qquad 5$$

where $\binom{i}{j} = \frac{i!}{j!*(i-j)!}$. The Bernoulli polynomials that will be employed in this paper is Eq 5. These are the first eight Bernoulli polynomials:

$B_o(z) = 1,$

$B_1(z) = z - \frac{1}{2},$

$B_2(z) = z^2 - z + \frac{1}{6},$

$B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z \;,$

$B_4(z) = z^4 - 2z^3 + z^2 - \frac{1}{30},$

$B_5(z) = z^5 - \frac{5}{2}z^4 - \frac{5}{3}z^3 + \frac{1}{6}\, z,$

$B_6(z) = z^6 - 3z^5 + \frac{5}{2}z^4 - \frac{1}{2}z^2 + \frac{1}{42},$

$B_7(z) = z^7 - \frac{7}{2}z^6 + \frac{7}{2}z^5 - \frac{7}{6}z^3 + \frac{1}{6}z \;,$

$B_8(z) = z^8 - 4z^7 + \frac{14}{3}z^6 - \frac{7}{3}z^4 + \frac{2}{3}z^2 - \frac{1}{30},$

and so on.

Description of the new technique:

The current section discusses introducing a new numerical approach to solving the integral equation with the singular or weakly singular kernel (Eq 1 or Eq 2), using the Bernoulli polynomial by assuming that:

$$u_M(x) = C^T B(x) \qquad 6$$

where C is a $(M+1)\times 1$ vector with unknown elements, M is the degree of the approximated polynomial, and $B(x)$ is the Bernoulli basis vector$(M+1)\times 1$ given by Eq 5. Now, substitute Eq 6 in Eq 1 or Eq 2 yields:

$$C^T B(x) = f(x) + \int_a^x K(x,t)(C^T B(t))dt \qquad 7$$

or,

$$C^T B(x) = f(x) + \int_a^b K(x,t)(C^T B(t))dt \qquad 8$$

which gives:

$$C^T B(x) - C^T \int_a^x K(x,t)B(t)dt = f(x) \qquad 9$$

or,

$$C^T B(x) - C^T \int_a^b K(x,t)B(t)dt = f(x) \qquad 10$$

assume that:

$$\alpha(x) = \int_a^x K(x,t)B(t)dt \text{ and } \beta(x) = \int_a^b K(x,t)B(t)dt \qquad 11$$

Substituting Eq 11 in Eq 9 and Eq 10, to obtain:

$$C^T[B(x) - \alpha(x)] - f(x) = 0 \qquad 12$$

or,

$$C^T[B(x) - \beta(x)] - f(x) = 0 \qquad 13$$

First, compute the integrals in Eq 11, to find the value of $\alpha(x)$ or $\beta(x)$, therefore collect the coefficients of like powers of x in Eq 12 or Eq 13. To derive a recurrence relation in $c_j, j \geq 0$, next compare the coefficients of like powers of x on both sides of the resulting equation. The coefficients $c_j, j \geq 0$ can be completely determined by solving the recurrence relation. The series solution is obtained by replacing the derived coefficients after determining the coefficients in Eq 6. In the section that follows, the test problems are discussed to demonstrate the efficacy of the suggested strategy.

Algorithm for the Proposed Approach:

To solve Eq 1 [or Eq 2] using the Bernoulli polynomial method, the following steps are followed:

Step 1: Assume that $f(x), k(x,t), a$ and b are given.

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Step 2: Define the degree of the Bernoulli polynomial M.

Step 3: Calculate the $\alpha_i(x)$ [or $\beta_i(x)$], $i = 0,1,\dots,M$ using Eq 11.

Step 4: Substituting the value of $\alpha(x)$[or $\beta(x)$], that obtained in step 3 in Eq 12 [or in Eq 13].

Step 5: Equating the coefficients of the powers of x to obtain C.

Step 6: Obtain the solution $u_M(x)$ by using the obtained value of C in Eq 6.

Convergence and Error Analysis:

The convergence of the proposed method can be obtained using the same manner as in [23], to conclude that: the sequence of error $\{\|u-u_M\|_2\}_{M=0}^{\infty} \to 0$ as $M \to \infty$.

Results and Discussion

Illustrative examples:

In this section, Bernoulli polynomial in the form $u_M(x) = C^T B(x)$ are used to solve integral equations with the singular or weakly singular kernel. For each example, two cases with M=4 and M=8 are employed. Also, a comparison with the analytic solution is made depending on the absolute errors to verify the accuracy and efficiency where:

$$\text{Absolute Errors} = \mathrm{E_M} = |u(x) - u_M(x)|.$$

With $u(x)$ and $u_M(x)$ are the exact and approximate solution respectively.

Test Example 1: Consider Abel's integral equation of the 1st kind[11]:

$$\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) \;, 0 \le x \le 1 \qquad 14$$

provided that the exact solution $u(x)=x^3 - x^2 + 1$. Utilizing the suggested technique for M= 4 and 8 the approximate solution is obtained as follows:

Case I: using M= 4.

First, substitute Eq 6 in Eq 14, which yields:

$$\int_0^x \frac{C^T B(t)}{\sqrt{x-t}} dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) \;,$$ and hence:

$$\int_0^x \frac{c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)}{\sqrt{x-t}} dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3).$$

Then,

$$c_0 \int_0^x \frac{B_0(t)}{\sqrt{x-t}}dt + c_1 \int_0^x \frac{B_1(t)}{\sqrt{x-t}}dt + c_2 \int_0^x \frac{B_2(t)}{\sqrt{x-t}}dt + c_3 \int_0^x \frac{B_3(t)}{\sqrt{x-t}}dt + c_4 \int_0^x \frac{B_4(t)}{\sqrt{x-t}}dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3).$$

Using Eq 11, to obtain:

$$c_0\alpha_0(x) + c_1\alpha_1(x) + c_2\alpha_2(x) + c_3\alpha_3(x) + c_4\alpha_4(x) = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3) \qquad 15$$

Such that: $\alpha_i(x) = \int_0^x \frac{B_i(t)}{\sqrt{x-t}}dt$ $\quad i = 0,1,2,3, and\ 4$. Computing $\alpha_i(x), i = 0,1,2,3\ and\ 4$, hence:

$$\alpha_0(x) = 2\sqrt[2]{x}, \alpha_1(x) = \frac{(4x-3)}{3}\sqrt[2]{x}\,, \alpha_2(x) = \frac{16\,x^2 - 20x + 5}{15}\sqrt[2]{x},$$

$$\alpha_3(x) = 2\,\frac{48\,x^2 - 84x + 35}{105}\sqrt[2]{x^3}, \;\; and\ \alpha_4(x) = \frac{256\,x^4 - 567\,x^3 + 336\,x^2 - 21}{315}\sqrt[2]{x}.$$

Substituting the value of $\alpha_i(x), i = 0,1,2,3$ and 4 in Eq 15, then equating the coefficients of the powers of x to obtain C^T:

$$C^T = \begin{bmatrix} 0.91666666666666666666666666666666, \\ -6.1875 \times 10^{-33}, 0.5, 1.0, 0 \end{bmatrix},$$

which leads to:

$$u_4(x) = x^3 - x^2 - 1.5000000356133419855416241048822 \times 10^{-33}\; x + 1.$$

Case II: using M= 8.

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T = \begin{bmatrix} 0.91666666666666666666666666666666, \\ (-6.1875)\times 10^{-33}, 0.5, 1.0, 0, 0, 0, 0, 0 \end{bmatrix},$$

and $u_8\,(x) = x^3 - x^2 + (1.5000000356133419855416241048822 \times 10^{-33})x + 1.$

Table 1 gives the absolute error calculated between the exact and approximated solution using the proposed method with M=4 and 8 for some points in the domain $[0,1]$ with $h = 0.1$. Also, Comparing the results obtained in the two cases with those obtained in[11] using the Bernoulli wavelet method concludes that the proposed method gives a better approximation as shown in Table 2.

Table 1. Absolute Error for M=4 and 8 of Test Example 1

x_i	E_4	E_8
0	0	0
0.1	1.50000003561334e-34	1.50000003561334e-34
0.2	3.00000007122668e-34	3.00000007122668e-34
0.3	4.50000010684003e-34	4.50000010684003e-34
0.4	6.00000014245337e-34	6.00000014245337e-34
0.5	7.50000017806671e-34	7.50000017806671e-34
0.6	9.00000021368005e-34	9.00000021368005e-34
0.7	1.05000002492934e-33	1.05000002492934e-33
0.8	1.20000002849067e-33	1.20000002849067e-33
0.9	1.35000003205201e-33	1.35000003205201e-33
1	1.50000003561334e-33	1.50000003561334e-33

Table 2. Maximum Absolute Error of Test Example 1

Absolute Error	**Bernoulli polynomial $N = 4, u_4(x_i)$**	**Bernoulli polynomial $N = 8, u_8(x_i)$**	**Bernoulli wavelet method[11]**
	1.500000035613342e-33	1.500000035613342e-33	1.33e-15

Test Example 2:Consider Abel's integral equation of the 2nd kind [4]:

$$u(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt \qquad 16$$

, provided that the exact solution $u(x)=x^2$. By using the proposed method for M= 4 and 8 the approximate solution is obtained as follows:

Case I: using M= 4

First, substitute Eq 6 in Eq 16, which yields:

$C^T B(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{C^T B(t)}{\sqrt{x-t}}dt$, and hence:

$$c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + c_4B_4(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)}{\sqrt{x-t}}dt.$$

Then, $c_0[B_0(x) + \int_0^x \frac{B_0(t)}{\sqrt{x-t}}dt] + c_1[B_1(x) + \int_0^x \frac{B_1(t)}{\sqrt{x-t}}dt] + c_2[B_2(x) + \int_0^x \frac{B_2(t)}{\sqrt{x-t}}dt] + c_3[B_3(x) + \int_0^x \frac{B_3(t)}{\sqrt{x-t}}dt] + c_4[B_4(x) + \int_0^x \frac{B_4(t)}{\sqrt{x-t}}dt] = x^2 + \frac{16}{15}x^{5/2}$.

Using Eq 11, to obtain:

$$c_0[B_0(x) + \alpha_0(x)] + c_1[B_1(x) + \alpha_1(x)] + c_2[B_2(x) + \alpha_2(x)] + c_3[B_3(x) + \alpha_3(x)] + c_4[B_4(x) + \alpha_4(x)] = x^2 + \frac{16}{15}x^{5/2} \qquad 17$$

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Such that $\alpha_i(x) = \int_0^x \frac{B_i(t)}{\sqrt{x-t}} dt \;, i = 0,1,2,3, \text{and } 4$. Computing $\alpha_i(x), i = 0,1,2,3$ and 4, to obtain:

$$\alpha_0(x) = 2\sqrt[2]{x}, \alpha_1(x) = \frac{(4x-3)}{3}\sqrt[2]{x}\,, \alpha_2(x) = \frac{16\,x^2 - 20x + 5}{15}\sqrt[2]{x},$$

$$\alpha_3(x) = 2\,\frac{48\,x^2 - 84x + 35}{105}\sqrt[2]{x^3}, \;\; and\; \alpha_4(x) = \frac{256\,x^4 - 567\,x^3 + 336\,x^2 - 21}{315}\sqrt[2]{x}.$$

Substituting the value of $\alpha_i(x), i = 0,1,2,3$ and 4 in Eq 17, then equating to the coefficients of the powers of x to obtain C^T:

$$C^T = [\,0.33333333333333333333333333333333, 1.0, 1.0, 0, 0],$$

which leads to: $u_4(x) = x^2 - 3.3333332594149330378849231885313 * 10^{-34}$.

Case II: using M= 8.

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T = [\,0.33333333333333333333333333333333, 1.0, 1.0, 0, 0, 0, 0, 0, 0],$$

and $u_8(x) = x^2 - 3.3333328002374522479288653882436 \times 10^{-34}$.

Table 3 gives a comparison between the exact and approximated solution using the proposed method with M=4 and 8 for some points in the domain [0,1] with $h = 0.1$ using absolute errors. Also, Comparing the results obtained in the two cases with those obtained in [4] using the Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet method concludes that the proposed method gives a better approximation as shown in Table 4:

Table 3. Absolute Error for M=4 and 8 of Test Example 2

x_i	E_4	E_8
0	3.33333325941493e-34	3.33333325941493e-34
0.1	3.33333325941493e-34	3.33333325941493e-34
0.2	3.33333325941493e-34	3.33333325941493e-34
0.3	3.33333325941493e-34	3.33333325941493e-34
0.4	3.33333325941493e-34	3.33333325941493e-34
0.5	3.33333325941493e-34	3.33333325941493e-34
0.6	3.33333325941493e-34	3.33333325941493e-34
0.7	3.33333325941493e-34	3.33333325941493e-34
0.8	3.33333325941493e-34	3.33333325941493e-34
0.9	3.33333325941493e-34	3.33333325941493e-34
1	3.33333325941493e-34	3.33333325941493e-34

Table 4. Maximum Absolute Error of Test Example 2

Absolute Error	**Bernoulli polynomial $N = 4, u_4(x_i)$**	**Bernoulli polynomial $N = 8, u_8(x_i)$**	**HOBW.[4] M=32,K=6**
	3.333333259414933e-34	3.333333259414933e-34	3.51 e−014

Test Example 3: Consider the following 2nd kind Weakly singular Fredholm integral equation [2]:

$$u(x) = x^2 - \frac{16}{15} + \int_0^1 \frac{u(t)}{\sqrt{1-t}} dt \qquad 18$$

Provided that the exact solution $u(x) = x^2$. By using the proposed method for N=4 and 8 the approximate solution is obtained as follows:

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Case I: using M= 4.

First, substitute Eq 6 in Eq 18, which yields:

$C^TB(x) = x^2 - \frac{16}{15} + \int_0^1 \frac{C^TB(t)}{\sqrt{1-t}}dt$, hence:

$$c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + c_4B_4(x) = x^2 - \frac{16}{15} + \int_0^1 \frac{c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)}{\sqrt{1-t}}dt.$$

Then $c_0[B_0(x) - \int_0^1 \frac{B_0(t)}{\sqrt{1-t}}dt] + c_1[B_1(x) - \int_0^1 \frac{B_1(t)}{\sqrt{1-t}}dt] + c_2[B_2(x) - \int_0^1 \frac{B_2(t)}{\sqrt{1-t}}dt] + c_3[B_3(x) - \int_0^1 \frac{B_3(t)}{\sqrt{1-t}}dt] + c_4[B_4(x) - \int_0^1 \frac{B_4(t)}{\sqrt{1-t}}dt] = x^2 - \frac{16}{15}$.

Using Eq 11, to obtain:

$$c_0[B_0(x) - \beta_0(x)] + c_1[B_1(x) - \beta_1(x)] + c_2[B_2(x) - \beta_2(x)] + c_3[B_3(x) - \beta_3(x)] + c_4[B_4(x) - \beta_4(x)] = x^2 - \frac{16}{15}. \quad 19$$

Such that $\beta_i(x) = \int_0^1 \frac{B_i(t)}{\sqrt{1-t}}dt$, $i = 0,1,2,3, \text{and } 4$. Computing $\beta_i(x), i = 0,1,2,3$ and 4, hence:

$$\beta_0(x) = 2,\ \beta_1(x) = \frac{1}{3},\ \beta_2(x) = \frac{1}{15},$$
$$\beta_3(x) = -\frac{2}{105} \text{ and } \beta_4(x) = \frac{-1}{63}.$$

Substituting the value of $\beta_i(x), i = 0,1,2,3$ and 4 in Eq. 19 then equating to the coefficients of the powers of x to obtain C^T:

$$C^T = [\ 0.33333333333333333333333333333334, 1, 1, 0, 0],$$

which leads to: $u_4(x) = x^2 + 3.66666667719192249966462706744 19 \times 10^{-33}$.

Case II: using M= 8.

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T = [\ 0.33333333333333333333333333333334, 1, 1, 0, 0, 0, 0, 0, 0],$$

and $u_8(x) = x^2 + 3.66666672310967057866023284747 07 \times 10^{-33}$.

Table 5 gives a comparison between the exact and approximated solution using the proposed method with M=4 and 8 for some points in the domain [0,1] with $h = 0.1$ using absolute errors. Also, Comparing the results obtained in the two cases with those obtained in[2] using the Euler–Maclaurin summation formula concludes that the proposed method gives a better approximation as shown in Table 6.

Table 5. Absolute Error for M=4 and 8 of Test Example 3

x_i	E_4	E_8
0	3.66666667719192e-33	3.66666672310967e-33
0.1	3.66666667719192e-33	3.66666672310967e-33
0.2	3.66666667719192e-33	3.66666672310967e-33
0.3	3.66666667719192e-33	3.66666672310967e-33
0.4	3.66666667719192e-33	3.66666672310967e-33
0.5	3.66666667719192e-33	3.66666672310967e-33
0.6	3.66666667719192e-33	3.66666672310967e-33
0.7	3.66666667719192e-33	3.66666672310967e-33
0.8	3.66666667719192e-33	3.66666672310967e-33
0.9	3.66666667719192e-33	3.66666672310967e-33
1	3.66666667719192e-33	3.66666672310967e-33

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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Table 6. Maximum Absolute Error of Test Example 3

Absolute Error	Bernoulli polynomial $N = 4, u_4(x_i)$	Bernoulli polynomial $N = 8, u_8(x_i)$	Euler-Maclaurin formulas [2] n= 256
	3.666666677191922e-33	3.666666723109671e-33	5.0214e−06

Test Example 4: Consider the following Weakly singular Fredholm integral equation [2]:

$$u(x) = e^x - 4.0602 + \int_0^1 \frac{u(t)}{\sqrt{1-t}}dt \quad ,0 \leq t \leq 1 \qquad 20$$

provided that the exact solution $u(x) = e^x$. By using the proposed method for N=4 and 8 the approximate solution is obtained as follows:

Case I: using M= 4.

First, substitute Eq 6 in Eq 20, which yields:

$C^T B(x) = e^x - 4.0602 + \int_0^1 \frac{C^T B(t)}{\sqrt{1-t}}dt$, therefore:

$$c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + c_4B_4(x) = e^x - 4.0602 + \int_0^1 \frac{c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)}{\sqrt{1-t}}dt.$$

Then, $c_0[B_0(x) - \int_0^1 \frac{B_0(t)}{\sqrt{1-t}}dt] + c_1[B_1(x) - \int_0^1 \frac{B_1(t)}{\sqrt{1-t}}dt] + c_2[B_2(x) - \int_0^1 \frac{B_2(t)}{\sqrt{1-t}}dt] + c_3[B_3(x) - \int_0^1 \frac{B_3(t)}{\sqrt{1-t}}dt] + c_4[B_4(x) - \int_0^1 \frac{B_4(t)}{\sqrt{1-t}}dt] = e^x - 4.0602.$

Using Eq 11, to obtain:

$$c_0[B_0(x) - \beta_0(x)] + c_1[B_1(x) - \beta_1(x)] + c_2[B_2(x) - \beta_2(x)] + c_3[B_3(x) - \beta_3(x)] + c_4[B_4(x) - \beta_4(x)] = e^x - 4.0602, \qquad 21$$

Such that $\beta_i(x) = \int_0^1 \frac{B_i(t)}{\sqrt{1-t}}dt \quad i = 0,1,2,3,$ and 4. Computing $\beta_i(x), i = 0,1,2,3$ and 4, hence:

$\beta_0(x) = 2,\ \beta_1(x) = \frac{1}{3}, \beta_2(x) = \frac{1}{15},\ \beta_3(x) = -\frac{2}{105}$ and $\beta_4(x) = \frac{-1}{63}$.

Substituting the value of $\beta_i(x), i = 0,1,2,3$ and 4 in Eq. 21 then equating to the coefficients of the powers of x to obtain C^T:

$$C^T = [\ 1.7239566137566137566137566137566,\ 1.7083333333333333333333333333,\ 0.83333333333333333333333333333333,\ 0.25, 0.041666666666666666666666666666667],$$

which leads to:

$$u_4(x) = 0.04166666667x^4 + 0.1666666667x^3 + 0.5x^2 + x + 1.007289947$$

Case II: using M= 8.

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T = [\ 1.7183263140454316924905160199278,\ 1.7182787698412698412698412698,\ 0.85912698412698412698412698412698,\ 0.28634259259259259259259259259259,\ 0.071527777777777777777777777777778,\ 0.014236111111111111111111111111,\ 0.0023148148148148148148148148148148,\ 0.00029761904761904761904761904761905,\ 0.000024801587301587301587301587301587],$$

and

$u_8(x) = 0.0000248015873x^8 + 0.0001984126984x^7 + 0.001388888889x^6 + 0.008333333333x^5 +$

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
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$0.04166666667x^4 + 0.1666666667x^3 + 0.5x^2 + x + 1.000044788$.

Table 7 gives a comparison between the exact and approximated solution using the proposed method with M=8 for some points in the domain [0,1] with $h = 0.1$ using maximum absolute errors (MSE). Also, Comparing the results obtained in the two cases with those obtained in [2] using the Euler–Maclaurin summation formula concludes that the proposed method gives a better approximation as shown in Table 8.

Table 7. Absolute Error for M=4 and 8 of Test Example 4

x_i	E_4	E_8
0	0.00728994700000007	4.47880000000733e-05
0.1	0.00728986225771945	4.47880000309532e-05
0.2	0.00728718884010224	4.47879988322984e-05
0.3	0.00726863942492397	4.47879450125230e-05
0.4	0.00719858269428176	4.47872498343305e-05
0.5	0.00700617630424693	4.47823402004411e-05
0.6	0.00657114661712310	4.47584742448357e-05
0.7	0.00570807287509055	4.46684983103492e-05
0.8	0.00414901852596447	4.43862560401655e-05
0.9	0.00152433586953740	4.36158180535793e-05
1	0.00265854808904517	4.17294186548285e-05

Table 8. Maximum Absolute Error of Test Example 4

method	**Bernoulli polynomial** $\boldsymbol{N = 4, u_4(x_i)}$	**Bernoulli polynomial** $\boldsymbol{N = 8, u_8(x_i)}$
M.S.E	0.007289947000000	4.478800003095322e-05

Test Example 5: Consider the VIE of 2nd kind with a weakly singular kernel [24]:

$$u(x) = f(x) + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} u(t)\, dt \qquad 22$$

, where $f(x) = 0.71428571x^3$,and exact solution u(x)=x^3 for $\mu = 0.5$. By using the proposed method for M= 4 and 8 the approximate solution is obtained as follows:

Case I: using M= 4

First, substitute Eq 6 in Eq 22, which yields:

$C^T B(x) = 0.71428571x^3 + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} C^T B(t)dt$, and hence:

$$c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + c_4B_4(x) = 0.71428571x^3 + \int_0^x \frac{t^{\mu-1}}{x^{\mu}} \big(c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)\big)dt.$$

Then,

$$c_0\left[B_0(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} B_0(t)dt\right] + c_1\left[B_1(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} B_1(t)dt\right] + c_2\left[B_2(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} B_2(t)dt\right] + c_3\left[B_3(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} B_3(t)dt\right] + c_4\left[B_4(x) - \int_0^x \frac{t^{\mu-1}}{x^{\mu}} B_4(t)dt\right] = 0.71428571x^3.$$

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Using Eq 11, to obtain:

$$c_0[B_0(x)-\alpha_0(x)]+c_1[B_1(x)-\alpha_1(x)]+c_2[B_2(x)-\alpha_2(x)]+c_3[B_3(x)-\alpha_3(x)]+c_4[B_4(x)-\alpha_4(x)]=x^2+\frac{16}{15}x^{\frac{5}{2}}. \qquad 23$$

Such that $\alpha_i(x)=\int_0^x \frac{t^{\mu-1}}{x^{\mu}}B_i(t)dt$ $i=0,1,2,3,$ and 4. Computing $\alpha_i(x), i=0,1,2,3$ and 4, to obtain:

$$\alpha_0(x)=2,\ \alpha_1(x)=\frac{2*x}{3}-\ 1\alpha_2(x)=\frac{2*x^2}{5}-\frac{2*x}{3}+\frac{1}{3},\alpha_3(x)=\frac{2*x^3}{7}-\frac{3*x^2}{5}+\frac{x}{3}\ \text{ and } \alpha_4(x)=\frac{2*x^4}{9}-\frac{4*x^3}{7}+\frac{2*x^2}{5}-\frac{1}{15}.$$

Substituting the value of $\alpha_i(x), i=0,1,2,3$ and 4 in Eq 23, then equating to the coefficients of the powers of x to obtain C^T:

$$C^T=[\,0.24999999849999998691174596388009,\ 0.99999999399999994764698,\ 1.49999999099999992147047578328 06,\ 0.99999999399999994764698385552037, 0],$$

which leads to:

$$u_4(x)=0.99999999399999994764698385552037x^3-1.4999998519423496695592009847671\times10^{-33}x-5.0000005778886207417614714832285\times10^{-34}.$$

Case II: using M= 8.

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T=[\,0.24999999849999998691174596388009,\ 0.99999999399999994764698385552,\ 1.4999999909999999214704757832806,\ 0.99999999399999994764698385552037,\ 0,0,0,0,0], \text{ and}$$

$$u_8(x)=0.99999999399999994764698385552037x^3-1.4999998519423496695592009847671\times10^{-33}x-5.0000005778886207417614714832285\times10^{-34}.$$

Table 9 gives a comparison between the exact and approximated solution using the proposed method with M=4 and 8 for some points in the domain [0,1] with $h=0.1$ using absolute errors. Also, Comparing the results obtained in the two cases with those obtained in [24] using non-smooth solutions concludes that the proposed method gives a better approximation as shown in Table 10:

Table 9. Absolute Error for M=4 and 8 of Test Example 5

x_i	E_4	E_8
0	5.00000057788862e-34	5.00000057788862e-34
0.1	6.00000005235302e-12	6.00000005235302e-12
0.2	4.80000004188241e-11	4.80000004188241e-11
0.3	1.62000001413531e-10	1.62000001413531e-10
0.4	3.84000003350593e-10	3.84000003350593e-10
0.5	7.50000006544127e-10	7.50000006544127e-10
0.6	1.29600001130825e-09	1.29600001130825e-09
0.7	2.05800001795708e-09	2.05800001795708e-09
0.8	3.07200002680474e-09	3.07200002680474e-09
0.9	4.37400003816535e-09	4.37400003816535e-09
1	6.00000005235302e-09	6.00000005235302e-09

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Table 10. Maximum Absolute Error of Test Example 5

Absolute Error	Bernoulli polynomial $N = 4, u_4(x_i)$	Bernoulli polynomial $N = 8, u_8(x_i)$	HOBW. [24] M=32,K=6
	6.000000052353016e-09	3.333333259414933e-34	3.51 e−014

Test Example 6: Consider the VIE of 2nd kind with a weakly singular kernel [25]:

$$u(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{u(t)}{(x-t)^{\frac{1}{4}}}dt \qquad 24$$

, where the exact solution is u(x)=$1 - 2x$. By using the proposed method for M= 4 and 8 the approximate solution is obtained as follows:

Case I: using M= 4

First, substitute Eq 6 in Eq 24, which yields:

$C^T B(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}} C^T B(t)dt$, and hence:

$$c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) + c_4B_4(x) = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} - \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}\big(c_0B_0(t) + c_1B_1(t) + c_2B_2(t) + c_3B_3(t) + c_4B_4(t)\big)dt.$$

Then,

$$c_0\left[B_0(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_0(t)dt\right] + c_1\left[B_1(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_1(t)dt\right] + c_2\left[B_2(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_2(t)dt\right] + c_3\left[B_3(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_3(t)dt\right] + c_4\left[B_4(x) + \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_4(t)dt\right] = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}}.$$

Using Eq. 11, to obtain:

$$c_0[B_0(x) + \alpha_0(x)] + c_1[B_1(x) + \alpha_1(x)] + c_2[B_2(x) + \alpha_2(x)] + c_3[B_3(x) + \alpha_3(x)] + c_4[B_4(x) + \alpha_4(x)] = 1 - 2x - \frac{32}{21}x^{\frac{7}{4}} + \frac{4}{3}x^{\frac{3}{4}} \qquad 25$$

Such that $\alpha_i(x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{4}}}B_i(t)dt$, $i = 0,1,2,3,$ and 4. Computing $\alpha_i(x), i = 0,1,2,3$ and 4, to obtain:

$$\alpha_0(x) = (4x^{\frac{3}{4}})/3,\ \alpha_1(x) = (2x^{\frac{3}{4}}(8x - 7))/21, \alpha_2(x) = (2x^{\frac{3}{4}}(192x^2 - 264x + 77))/693, \alpha_3(x) = (2x^{\frac{3}{4}}(256x^3 - 480x^2 + 220x))/1155,$$

$$\text{and } \alpha_4(x) = (2x^{3/4}(12288x^4 - 29184x^3 + 18240x^2 - 1463))/65835$$

Substituting the value of $\alpha_i(x), i = 0,1,2,3$ and 4 in Eq. 25, then equating to the coefficients of the powers of x to obtain C^T:

$$C^T = [\,0, -2.0, 0, 0, 0],$$

which leads to: $u_4(x) = 1 - 2x$.

Case II: using M= 8.

Using the same manner as in case I, for M=8, the following C^T and approximate solutions are obtained:

$$C^T = [\,0, -2.0, 0, 0, 0],$$

and $u_8(x)$=$1 - 2x$.

Table 11 gives a comparison between the exact and approximated solution using the proposed method with M=4 and 8 for some points in the domain [0,1] with $h = 0.1$ using absolute errors. Also, Comparing the results obtained in the two cases with those obtained in [25] using non-smooth solutions concludes that the proposed method gives a better approximation as shown in Table 12:

Table 11. Absolute Error for M=4 and 8 of Test Example 6

x_i	E_4	E_8
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1	0	0

Table 12. Maximum Absolute Error of Test Example 6

Absolute Error	**Bernoulli polynomial** $N = 4, u_4(x_i)$	**Bernoulli polynomial** $N = 8, u_8(x_i)$	**Taylor collocation** N=8[25]
	0	0	1.28571×10^{-8}

Conclusion

In this study, integral equations with singular or weakly singular kernels are solved using Bernoulli's polynomial approach. The Bernoulli polynomial approach was used to resolve several associated examples. The numerical data and tables that are presented demonstrate the method's effectiveness and precision. noticed that when increasing the degree of the proposed technique, the error decreases. In addition, the numerical results showed that the Bernoulli method is capable of resolving the problems and identifying an accurate solution for a large number of integral equations with singular or weakly singular kernels, and comparisons demonstrate the superiority of the current approach. Therefore, it is recommended to apply this technique for finding the numerical solutions to other types of problems like fractional integro-differential equations or delay differential equations.

Acknowledgment

The authors are grateful for the referees' helpful suggestions and insightful comments, which helped to improve the manuscript significantly.

Authors' Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Published Online First: May, 2024
https://doi.org/10.21123/bsj.2024.9712
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Authors' Contribution Statement

The first author proposed and presented the idea of this method. The first author developed the paper through continuous discussions with the second author. The second author proposed the program and modifications of the method and presentation style of the draft paper. The final copy was prepared through discussion by both authors.

Journal Declaration:

M.M. M. is an editor for the journal but did not participate in the peer review process other than as an author. The authors declare no other conflict of interest.

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طريقة متعددات حدود برنولي لحل المعادلات التكاملية مع نواة مفردة

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قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق.

الخلاصة

هناك دائما حاجة إلى طريقة فعالة لتوليد حل عددي أكثر دقة للمعادلات التكاملية ذات النواة المفردة أو المفردة الضعيفة لأن الطرق العددية لها محدودة. في هذه الدراسة ، تم حل المعادلات التكاملية ذات النواة المفردة أو المفردة الضعيفة باستخدام طريقة متعددة حدود برنولي. الهدف الرئيسي من هذه الدراسة هو ايجاد حل تقريبي لمثل هذه المشاكل في شكل متعددة الحدود في سلسلة من الخطوات المباشرة. أيضا ، تم افتراض أن مقام النواة لن يكون صفرا أبدا أو أن يكون له قيمة عقدية بسبب اختيارالعقد المحددة لمتغيري النواة الوحيدين. مع متعددات حدود برنولي من الدرجة 4 و 8 كمثال على ذلك، يوفر النهج الحالي حلا قريبا جدا من الحل الدقيق في أمثلة الاختبار. بينما. يثبت الحجم المتواضع جدا للأخطاء في أمثلة الاختبار فعالية الاستراتيجية الحالية. أيضا ، فإن السهولة التي يمكن بها تنفيذ برنامج الكمبيوتر تجعل هذه التقنية فعالة للغاية. هدف آخر هو تحديد كفاءة الطريقة المقترحة من خلال مقارنتها بأساليب مختلفة. يظهر أن الحل التقريبي للمعادلات التكاملية ذات النواة المفردة أو المفردة الضعيفة يتقارب بشدة مع الحل المضبوط للمعادلات باستخدام متعددة حدود برنولي وهو متفوق على تلك الموجودة في الأساليب الأخرى المذكورة. هذا يضمن الأصالة والدقة العالية للطريقة المقترحة. كذلك تمت مناقشة تقارب الحل. تم تنفيذ البرامج باستخدام برنامج ال MATLAB النسخة 2018a .

الكلمات المفتاحية: معادلة ابل التكاملية ، متعددة حدود برنولولي، معادلة تكاملية، نواة مفردة، نواة مفردة ضعيفة.