Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$\Bbbk$ –Mixed Modulus of Smoothness

Rehab Amer Kamel[1], **Eman Samir Bhaya** [2,3]*

[1]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hillah, Iraq.
[2]Department of Mathematics, College of Education, Al-Zahraa University for Women, Karbala, Iraq.
[3]Department of Mathematics, College of Education for Pure Sciences, University of Babylon, Hillah, Iraq.
*Corresponding Author

Received 28/09/2023, Revised 01/03/2024, Accepted 03/03/2024, Published Online First 20/07/2024

Abstract

The modulus of smoothness is essential for modern analysis and its applications. Is a versatile tool in approximation theory that helps in understanding the properties of approximation methods, characterizing function spaces, and analyzing the convergence and accuracy of numerical algorithms. It helps in determining the optimal number of terms or the optimal choice of basic functions to achieve the desired level of accuracy in approximation of a function and rate of convergence. The smoothness modulus has many applications, including its applications in numerical analysis, particularly in the analysis of numerical methods for solving differential equations, optimization problems, and integral equations. Many papers introduced the ordinary modulus of smoothness with one variable. However, few researchers have tried to approximate functions with multiple variables and mixed modulus. This paper tries to fill in that gap introducing a new $\Bbbk$ -mixed modulus of smoothness for measurable functions $f \in L_p([0,2\pi]^d), 0 < p < 1$ with d –variables. It does this by using a new $\Bbbk$ –mixed difference and proving some of its approximation properties, like linearity and monotonicity, for the $\Bbbk$ -mixed modulus of smoothness of functions belonging to the space L_p using the vector of numbers $k_i, i = 1, \dots, d$. Also, study the approximation of bounded functions with $\Bbbk$ -mixed difference and its direct relationship with mixed smoothness.

Keywords: Mixed Derivatives, Mixed Finite Differences, Mixed Modulus of Smoothness, Monotonicity Property, Space of Measurable Functions.

Introduction

Many studies have appeared that dealt with the approximation theory in general, such as[1,2], and the modulus of smoothness, in particular, is a powerful tool in approximation theory used to investigate the rate of convergence of approximation methods. The first to develop a concept of dominant mixed modulus of smoothness was Nikol'skii[3], who defined the Sobolev spaces $W_p^\alpha(\Omega)$ defined on an open set $\Omega \subset R^n$, and it consists of functions that can be p-integrated on Ω up to order α in their partial derivatives. Here, α is a non-negative integer, and $1 < p < \infty$. The space is equipped with a norm that measures the size of the function and its derivatives. The norm of a function $g \in W_p^\alpha(\Omega)$ is given as follows:

$$\|g\|_{W_p^\alpha(\Omega)} = \left(\sum_{|\alpha|} \|D^\alpha g\|_{L_p(\Omega)}^p\right)^{1/p} \qquad 1$$

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

where α is multi-index, D^{α} denotes the partial derivative of order $|\alpha|$,and $L_p(\Omega)$ is the $p-$th Lebesgue space. A dominating mix of smoothness comes from the mixed derivative. Subsequently, studies continued on the dominant mixed smoothness; for example see[4-7]. The researchers were also interested in studying the mixed modulus of smoothness it is between mixed metric spaces like[8,9].

In the context of modulus smoothness, precisely that the function g has r-th symmetric difference, according to what follows:

$$\Delta_h^{r} g(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} g(x + jh). \qquad 2$$

where, the function $g \in L_p$ has the r -th usual modulus of smoothness which is defined by

$$\omega_r(g, \delta)_p = \sup_{0<h\leq\delta} \|\Delta_h^{r}(g,.)\|_p,\ \delta \geq 0 \qquad 3$$

From Eq 2 and Eq 3, it can be developed modulus not in the usual manner, but by defining the modulus of a $\mathbb{k}$ -mixed smoothness of a function $f \in L_p$ as follows:

$$\overline{\omega_{\mathbb{k}}}(f, \delta)_p = \sup_{|h_i|\leq\delta}{}_{i=1}^{d} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1, x_2, \dots, x_i, \dots, x_d) \right\|_p,\ \delta > 0 \text{ , or} \qquad 4$$

$$\underline{\omega_{\mathbb{k}}}(f, \delta)_p = \sup_{|h_i|\leq\delta}{}_{i=1}^{d} \left\| \underline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1, x_2, \dots, x_i, \dots, x_d) \right\|_p$$

Then, from Eq.4, can be defined $\mathbb{k}$ –mixed difference as mentioned below:

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(\mathbb{x}) = \sup_{i=1}^{d} \left\{\Delta_{h_i}^{k_i} f(\mathbb{x})\right\} \qquad 5$$

$$= \sup_{i_{1\leq i\leq d}} \left\{\Delta_{h_1}^{k_1} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{k_d} f(\mathbb{x})\right\}.$$

Or

$$\underline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(\mathbb{x}) = \inf_{i=1}^{d} \left\{\Delta_{h_i}^{k_i} f\mathbb{x}\right\} \qquad 6$$

$$= \inf_{i_{1\leq i\leq d}} \left\{\Delta_{h_1}^{k_1} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{k_d} f\mathbb{x}\right\},$$

where $\mathbb{k} = (k_1, k_2, \dots, k_i, \dots, k_d)$, $\mathbb{h} = (h_1, h_2, \dots, h_i, \dots, h_d)$, $\mathbb{x} = (x_1, x_2, \dots, x_i, \dots, x_d)$.

$L_p = L_p([0,2\pi]^d)$, $0 < p < 1$, be the measurable function's space,and 2π -periodic in all d -variables so that:

$$\|f(\mathbb{x})\|_p = \left(\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} |f(x_1, x_2, \dots, x_i, \dots, x_d)|^p dx_1 dx_2 \dots dx_i \dots dx_d\right)^{\frac{1}{p}}. \qquad 7$$

Also, $\Delta_{h_i}^{k_i}$ is the standard difference in variable x_i of order k_i with step h_i ; for instance,

$$\Delta_{h_i}^{1} f(\mathbb{x}) = \Delta_{h_i}^{1} f(x_1, \dots, x_i, \dots, x_d) = f(x_1, \dots, x_i + h_i, \dots, x_d) - f(x_1, \dots, x_i, \dots, x_d),$$

$$\Delta_{h_i}^{k_i} f(\mathbb{x}) = \Delta_{h_i}^{k_i} f(\mathbb{x})$$

$$= \sum_{j=0}^{k_i} (-1)^j \binom{k_i}{j} f(x_1, \dots, x_i + (k_i - j)h_i, \dots, x_d),\ k_i \in \mathbb{N}. \qquad 8$$

Using Eq 5 and Eq 6 to obtain

$$\Delta_{\gamma\,\mathbb{h}}^{\mathbb{k}} f(\mathbb{x}) = \sum_{j_{k_1}=0}^{\gamma-1} \dots \sum_{j_{k_i}=0}^{\gamma-1} \dots \sum_{j_{k_d}=0}^{\gamma-1} \sup_{i=1}^{d} \left\{\Delta_{h_i}^{k_i} f(x_1, \dots, x_d)\right\} \qquad 9$$

$$\Delta_{\gamma\,\mathbb{h}}^{\mathbb{k}} f(\mathbb{x}) = \sum_{j_{k_1}=0}^{\gamma-1} \dots \sum_{j_{k_i}=0}^{\gamma-1} \dots \sum_{j_{k_d}=0}^{\gamma-1} \sup_{i=1}^{d} \left\{\Delta_{h_1}^{k_1} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{k_d} f(\mathbb{x})\right\}$$

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

Materials and Methods

Note: All the following results $\underline{\Delta_{\mathbb{h}}^{\mathbb{k}}}f(\mathbb{x})$ and $\underline{\omega_{\mathbb{k}}}(f,\delta)_p$ are similar to $\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,f(\mathbb{x})$ and $\overline{\omega_{\mathbb{k}}}(f,\delta)_p$ in terms of proof.

Lemma 1:[10]

Suppose that $\langle \mathfrak{x}_l \rangle$ be any sequence; then

$|\sum_l \mathfrak{x}_l|^p \leq 2^{\frac{1}{p}} \sum_l |\mathfrak{x}_l|^p$ for $1 \leq p < \infty$;

$|\sum_l \mathfrak{x}_l|^p \leq \sum_l |\mathfrak{x}_l|^p$ for $0 < p < 1$.

Lemma 2:[11]

For $0 < p < 1$, f and g functions $\in L_p([0,2\pi]^d)$, then

$$\|(f+g)(\mathbb{x})\|_p \leq (2)^{\frac{1}{p}-1}\|f(\mathbb{x})\|_p + (2)^{\frac{1}{p}-1}\|g(\mathbb{x})\|_p.$$

Lemma 3:

Let f and g $\in L_p([0,2\pi]^d), 0 < p < 1$. Then

(a) $\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,(f+g)(\mathbb{x}) = \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,f(\mathbb{x}) + \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,g(\mathbb{x})$;

(b) $\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\overline{\Delta_{\mathbb{h}}^{\mathbb{r}}}\,f(\mathbb{x})\right) = \overline{\Delta_{\mathbb{h}}^{\mathbb{k}+\mathbb{r}}}\,f(\mathbb{x})$;

(c) $\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,f(\mathbb{x})\right\|_p \leq C(p,k_l)\|f(\mathbb{x})\|_p$;

(d) $\lim_{\mathbb{h}\to 0^+}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,f(\mathbb{x})\right\|_p = \lim_{h_l\to 0^+}\left\|\Delta_{h_l}^{k_l}\,f(\mathbb{x})\right\|_p = 0.$

Proof of:

(a)

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,(f+g)(\mathbb{x}) = \sup_{i=1}^{d}\left\{\Delta_{h_i}^{k_i}\,(f+g)(\mathbb{x})\right\}$$

$$= \sup_{i=1}^{d}\left\{\Delta_{h_1}^{k_1}\,(f+g)(\mathbb{x}), \Delta_{h_2}^{k_2}\,(f+g)(\mathbb{x}), \dots, \Delta_{h_i}^{k_i}\,(f+g)(\mathbb{x}), \dots, \Delta_{h_d}^{k_d}\,(f+g)(\mathbb{x})\right\}$$

$$= \sup_{i=1}^{d}\left\{\begin{array}{c} \sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}\,(f+g)(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}\,(f+g)(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}\,(f+g)(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}\,(f+g)(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{array}\right\}$$

From the properties of finite difference, the following can be concluded

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,(f+g)(\mathbb{x}) = \sup_{i=1}^{d}\left\{\begin{array}{c} \sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}\,f(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d) + \\ \sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}\,g(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}\,f(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d) + \\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}\,g(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}\,f(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d) + \\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}\,g(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}\,f(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) + \\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}\,g(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{array}\right\}$$

Then from properties of supremum,

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\,(f+g)(\mathbb{x}) = \sup_{i=1}^{d}\left\{\begin{matrix} \sum_{j=0}^{k_1}(-1)^j \binom{k_1}{j}\; f(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j \binom{k_2}{j}\; f(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j \binom{k_i}{j}\; f(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j \binom{k_d}{j}\; f(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{matrix}\right\}$$

$$+\sup_{i=1}^{d}\left\{\begin{matrix} \sum_{j=0}^{k_1}(-1)^j \binom{k_1}{j}\; g(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j \binom{k_2}{j}\; g(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j \binom{k_i}{j}\; g(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j \binom{k_d}{j}\; g(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{matrix}\right\}$$

$$= \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\, f(\mathbb{x}) + \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\, g(\mathbb{x})\ .$$

(b) $\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\overline{\Delta_{\mathbb{h}}^{\mathbb{r}}}\, f(\mathbb{x})\right) = \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\sup_{i=1}^{d}\left\{\Delta_{h_i}^{r_i}\, f(\mathbb{x})\right\}\right)$

$$= \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right)$$

$$= \sup_{i=1}^{d}\left\{\Delta_{h_i}^{k_i}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right)\right\}$$

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\overline{\Delta_{\mathbb{h}}^{\mathbb{r}}}\, f(\mathbb{x})\right) = \sup_{i=1}^{d}\left\{\begin{matrix} \Delta_{h_1}^{k_1}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right), \\ \Delta_{h_2}^{k_2}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right), \\ \dots, \\ \Delta_{h_i}^{k_i}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right), \\ \dots, \\ \Delta_{h_d}^{k_d}\left(\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right) \end{matrix}\right\}$$

Choose $r_1, r_2, \dots, r_i, \dots, r_d$ s.t. $\Delta_{h_1}^{r_1} f(\mathbb{x}) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}$

$$\Delta_{h_2}^{r_2} f(\mathbb{x}) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}$$

$\vdots$

$$\Delta_{h_i}^{r_i} f(\mathbb{x}) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}$$

$\vdots$

$$\Delta_{h_d}^{r_d} f(\mathbb{x}) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}$$

Then

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\overline{\Delta_{\mathbb{h}}^{\mathbb{r}}}\, f(\mathbb{x})\right) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{k_1}\left(\Delta_{h_1}^{r_1} f(\mathbb{x})\right), \Delta_{h_2}^{k_2}\left(\Delta_{h_2}^{r_2} f(\mathbb{x})\right), \dots, \Delta_{h_i}^{k_i}\left(\Delta_{h_i}^{r_i} f(\mathbb{x})\right), \dots, \Delta_{h_d}^{k_d}\left(\Delta_{h_d}^{r_d} f(\mathbb{x})\right)\right\}$$

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

From properties of finite difference, the following result

$$\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\left(\overline{\Delta_{\mathbb{h}}^{\mathbb{r}}}\, f(\mathbb{x})\right) = \sup_{i=1}^{d}\left\{\Delta_{h_1}^{k_1+r_1} f(\mathbb{x}), \Delta_{h_2}^{k_2+r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i+r_i} f(\mathbb{x}), \dots, \Delta_{h_1}^{k_1+r_1} f(\mathbb{x}), \Delta_{h_d}^{k_d+r_d} f(\mathbb{x})\right\}$$

$$= \overline{\Delta_{\mathbb{h}}^{\mathbb{k}+\mathbb{r}}}\, f(\mathbb{x}).$$

(c) $\left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\, f(\mathbb{x})\right\|_p = \left\|\sup_{i=1}^{d}\left\{\Delta_{h_i}^{k_i} f(\mathbb{x})\right\}\right\|_p$

$$= \left\|\sup_{i=1}^{d}\left\{\Delta_{h_1}^{r_1} f(\mathbb{x}), \Delta_{h_2}^{r_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{r_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{r_d} f(\mathbb{x})\right\}\right\|_p$$

$$= \left\| \sup_{i=1}^{d} \left\{ \begin{array}{c} \sum_{j=0}^{k_1}(-1)^j \binom{k_1}{j} f(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j \binom{k_2}{j} f(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j \binom{k_i}{j} f(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j \binom{k_d}{j} f(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{array} \right\} \right\|_p$$

$$\left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\, f(\mathbb{x})\right\|_p = \left(\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} \left| \sup_{i=1}^{d} \left\{ \begin{array}{c} \sum_{j=0}^{k_1}(-1)^j \binom{k_1}{j} f(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j \binom{k_2}{j} f(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j \binom{k_i}{j} f(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j \binom{k_d}{j} f(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{array} \right\} \right|^p dx_1 \dots dx_i \dots dx_d \right)^{\frac{1}{p}}$$

Choose k_l s.t. $\overline{\Delta_{h_l}^{k_l}}\, f(\mathbb{x}) = \sup_{i=1}^{d} \left\{ \begin{array}{c} \sum_{j=0}^{k_1}(-1)^j \binom{k_1}{j} f(x_1+(k_1-j)h_1, x_2, \dots, x_i, \dots, x_d), \\ \sum_{j=0}^{k_2}(-1)^j \binom{k_2}{j} f(x_1, x_2+(k_2-j)h_2, \dots, x_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_i}(-1)^j \binom{k_i}{j} f(x_1, x_2, \dots, x_i+(k_i-j)h_i, \dots, x_d), \dots, \\ \sum_{j=0}^{k_d}(-1)^j \binom{k_d}{j} f(x_1, x_2, \dots, x_i, \dots, x_d+(k_d-j)h_d) \end{array} \right\}$

From Lemma 1

$\left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}\, f(\mathbb{x})\right\|_p \le \left(\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\sum_{j=0}^{k_l}\left|\binom{k_l}{j}\right|^p |f(x_1, x_2, \dots, x_l+(k_l-j)h_l, \dots, x_d)|^p\, dx_1 \dots dx_l \dots dx_d\right)^{\frac{1}{p}}$

$= \left(\sum_{j=0}^{k_l}\left|\binom{k_l}{j}\right|^p \int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}|f(x_1, x_2, \dots, x_l+(k_l-j)h_l, \dots, x_d)|^p\, dx_1 \dots dx_l \dots dx_d\right)^{\frac{1}{p}}$

$\le C(p, k_l)\left(\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}|f(x_1, x_2, \dots, x_l+(k_l-j)h_l, \dots, x_d)dx_1 \dots dx_l \dots dx_d|^p\right)^{\frac{1}{p}}.$

$\le C(p, k_l)\|f(\mathbb{x})\|_p\,.$

(d)

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$$\lim_{\mathbb{h}\to 0^+} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(\mathbb{x}) \right\|_p = \lim_{\mathbb{h}\to 0^+} \left\| \sup_{i=1}^{d} \left(\Delta_{h_i}^{k_i} f(\mathbb{x}) \right) \right\|_p$$

$$= \lim_{\mathbb{h}\to 0^+} \left\| \sup_{i=1}^{d} \left(\Delta_{h_1}^{k_1} f(\mathbb{x}), \Delta_{h_2}^{k_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{k_d} f(\mathbb{x}) \right) \right\|_p,$$

where $k_l \in \mathbb{N}$ such that $\Delta_{h_l}^{k_l} f(\mathbb{x}) = \sup_{i=1}^{d} \left\{ \Delta_{h_1}^{k_1} f(\mathbb{x}), \Delta_{h_2}^{k_2} f(\mathbb{x}), \dots, \Delta_{h_i}^{k_i} f(\mathbb{x}), \dots, \Delta_{h_d}^{k_d} f(\mathbb{x}) \right\}$

$$= \lim_{h_l\to 0^+} \left\| \Delta_{h_l}^{k_l} f(\mathbb{x}) \right\|_p$$

$$= \lim_{h_l\to 0^+} \left\| \sum_{j=0}^{k_l} (-1)^j \binom{k_l}{j} f(x_1, x_2, \dots, x_l + (k_l - j)h_l, \dots, x_d) \right\|_p$$

$$\Delta_{h_l}^{k_l} f(\mathbb{x}) = \lim_{h_l\to 0^+} \left\| \left((-1)^0 \frac{k_l!}{0!(k_l-0)!} f(x_1, x_2, \dots, x_l + k_l h_l, \dots, x_d) + (-1)^1 \frac{k_l!}{1!(k_l-1)!} f(x_1, x_2, \dots, x_l + (k_l-1)h_l, \dots, x_d) + \dots + (-1)^j \frac{k_l!}{j!(k_l-j)!} f(x_1, x_2, \dots, x_l + (k_l - j)h_l, \dots, x_d) + \dots + (-1)^{k_l} \frac{k_l!}{k_l!(k_l-k_l)!} f(x_1, x_2, \dots, x_l + (k_l - k_l)h_l, \dots, x_d) \right) \right\|_p$$

$$= \lim_{k_l\to\infty} \left\| \sum_{j=0}^{k_l} (-1)^j \binom{k_l}{j} f(x_1, x_2, \dots, x_l, \dots, x_d) \right\|_p = 0$$

Results and Discussion

In the following result, can be summarized the major characteristics of modulus of a $\mathbb{k}$ -mixed smoothness of $f \in L_p([0,2\pi]^d), 0 < p < 1$

Theorem 1:

Let f and $g \in L_p([0,2\pi]^d), 0 < p < 1$, and $\mathbb{k} = (k_1, k_2, \dots, k_i, \dots, k_d), k_i \in \mathbb{N}$. Then

(a) $\lim_{\delta\to 0^+} \overline{\omega_{\mathbb{k}}}(f, \delta)_p = 0$;

(b) $\overline{\omega_{\mathbb{k}}}(f+g, \delta)_p \le 2^{\frac{1}{p}-1} \left(\overline{\omega_{\mathbb{k}}}(f, \delta)_p + \overline{\omega_{\mathbb{k}}}(g, \delta)_p \right)$;

(c) $\overline{\omega_{\mathbb{k}}}(f, \delta)_p \le \overline{\omega_{\mathbb{k}}}(f, t)_p$, for $0 < \delta < t$;

(d) $\frac{\overline{\omega_{\mathbb{k}}}(f,\delta)_p}{\delta^{\mathbf{k}}} \le \frac{\overline{\omega_{\mathbb{k}}}(f,t)_p}{t^{\mathbf{k}}}$, for $0 < t \le \delta \le 1$;

(e) $\overline{\omega_{\mathbb{k}}}(f, \gamma\delta)_p \le \gamma^{\mathbf{k_l}} \omega_{\mathbf{k_l}}(f, \delta)_p$, for $\gamma > 1$ and $\gamma \in \mathbb{N}$, where $\mathbf{k_l} = \max|k_i|$, $i = 1, \dots, d$

Proof of:

(a)

$$\overline{\omega_{\mathbb{k}}}(f, \delta)_p = \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1, x_2, \dots, x_i, \dots, x_d) \right\|_p$$

$$\lim_{\delta\to 0^+} \overline{\omega_{\mathbb{k}}}(f, \delta)_p = \lim_{\delta\to 0^+} \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1, x_2, \dots, x_i, \dots, x_d) \right\|_p$$

From Lemma 3(d)

$$\lim_{\delta\to 0^+} \overline{\omega_{\mathbb{k}}}(f, \delta)_p = 0 .$$

(b)

$$\overline{\omega_{\mathbb{k}}}(f+g, \delta)_p = \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} (f+g)(\mathbb{x}) \right\|_p$$

$$= \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left\| \overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} (f+g)(x_1, \dots, x_d) \right\|_p$$

$$= \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left\| \sup_{i=1}^{d} \left(\Delta_{h_i}^{k_i} (f+g) \right) (x_1, \dots, x_d) \right\|_p$$

$$= \sup_{|h_i|\le\delta} {}_{i=1}^{d} \left(\int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \left| \sup_{i=1}^{d} \left\{ \Delta_{h_1}^{k_1} (f+g)(x_1, \dots, x_d), \Delta_{h_2}^{k_2} (f+g)(x_1, \dots, x_d), \dots, \Delta_{h_i}^{k_i} (f+g)(x_1, \dots, x_d), \dots, \Delta_{h_d}^{k_d} (f+g)(x_1, \dots, x_d) \right\} \right|^p dx_1 \dots dx_d \right)^{\frac{1}{p}}$$

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$$\overline{\omega_{\mathbb{k}}}(f+g,\delta)_p = \sup_{|h_i|\le\delta}{}_{i=1}^{d}\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\sup_{i=1}^{d}\left\{\begin{array}{l}\sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}(f+g)(x_1+(k_1-j)h_1,x_2,\dots,x_i,\dots,x_d),\\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}(f+g)(x_1,x_2+(k_2-j)h_2,\dots,x_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}(f+g)(x_1,x_2,\dots,x_i+(k_i-j)h_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}(f+g)(x_1,x_2,\dots,x_i,\dots,x_d+(k_d-j)h_d)\end{array}\right\}\right|^p dx_1\dots dx_d\right)^{\frac{1}{p}}$$

Can be obtained from Lemma 2 and Lemma 3(a),

$$\overline{\omega_{\mathbb{k}}}(f+g,\delta)_p \le 2^{\frac{1}{p}-1}\sup_{|h_i|\le\delta}{}_{i=1}^{d}\left(\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\sup_{i=1}^{d}\left\{\begin{array}{l}\sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}f(x_1+(k_1-j)h_1,x_2,\dots,x_i,\dots,x_d),\\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}f(x_1,x_2+(k_2-j)h_2,\dots,x_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}f(x_1,x_2,\dots,x_i+(k_i-j)h_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}f(x_1,x_2,\dots,x_i,\dots,x_d+(k_d-j)h_d)\end{array}\right\}\right|^p dx_1\dots dx_i\dots dx_d\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\sup_{i=1}^{d}\left\{\begin{array}{l}\sum_{j=0}^{k_1}(-1)^j\binom{k_1}{j}g(x_1+(k_1-j)h_1,x_2,\dots,x_i,\dots,x_d),\\ \sum_{j=0}^{k_2}(-1)^j\binom{k_2}{j}g(x_1,x_2+(k_2-j)h_2,\dots,x_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_i}(-1)^j\binom{k_i}{j}g(x_1,x_2,\dots,x_i+(k_i-j)h_i,\dots,x_d),\dots,\\ \sum_{j=0}^{k_d}(-1)^j\binom{k_d}{j}g(x_1,x_2,\dots,x_i,\dots,x_d+(k_d-j)h_d)\end{array}\right\}\right|^p dx_1\dots dx_i\dots dx_d\right)^{\frac{1}{p}}\right)$$

$$\overline{\omega_{\mathbb{k}}}(f+g,\delta)_p \le 2^{\frac{1}{p}-1}\left(\sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}f(x_1,x_2,\dots,x_i,\dots,x_d)\right\|_p + \sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}}g(x_1,x_2,\dots,x_i,\dots,x_d)\right\|_p\right)$$

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

$\overline{\omega_{\mathbb{k}}}(f+g,\delta)_p \le$
$2^{\frac{1}{p}-1}\left(\sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1,\dots,x_d)\right\|_p + \sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} g(x_1,\dots,x_d)\right\|_p\right)$

$\le 2^{\frac{1}{p}-1}(\overline{\omega_{\mathbb{k}}}(f,\delta)_p + \overline{\omega_{\mathbb{k}}}(g,\delta)_p)$.

(c) $\overline{\omega_{\mathbb{k}}}(f,\delta)_p = \sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(\mathbb{x})\right\|_p$

$\overline{\omega_{\mathbb{k}}}(f,\delta)_p = \sup_{|h_i|\le\delta}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1,x_2,\dots,x_i,\dots,x_d)\right\|_p$

since $0<\delta<t$, then

$\overline{\omega_{\mathbb{k}}}(f,\delta)_p \le \sup_{|h_i|\le t}{}_{i=1}^{d}\left\|\overline{\Delta_{\mathbb{h}}^{\mathbb{k}}} f(x_1,x_2,\dots,x_i,\dots,x_d)\right\|_p$

$= \overline{\omega_{\mathbb{k}}}(f,t)_p$.

(d) Suppose that $\mathbf{k} = \max|k_i|_{1\le i\le d}$

Since $\le \delta \le 1$, then $\frac{1}{\delta^k} \le \frac{1}{t^k}$.

From (c) $\rightarrow \frac{\overline{\omega_{\mathbb{k}}}(f,\delta)_p}{\delta^k} \le \frac{\overline{\omega_{\mathbb{k}}}(f,t)_p}{t^k}$, for $0<\delta<t$.

(e) By applying Eq 9, to obtain

$$\Delta_{\gamma\,\mathbb{h}}^{\mathbb{k}} f(\mathbb{x}) = \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_i}=0}^{\gamma-1}\dots\sum_{j_{k_d}=0}^{\gamma-1}\sup_{i=1}^{d}\left\{\Delta_{h_1}^{k_1} f(\mathbb{x}),\dots,\Delta_{h_i}^{k_i} f(\mathbb{x}),\dots,\Delta_{h_d}^{k_d} f(\mathbb{x})\right\}$$

,

where $\mathbf{k}_l = \max|k_i|$, $i=1,\dots,d.$, k_l is one component of $(k_1,\dots,k_d)$,

then

$\Delta_{\gamma h_l}^{\mathbf{k}_l} f(\mathbb{x}) = \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\Delta_{h_l}^{\mathbf{k}_l} f(x_1,\dots,x_l,\dots,x_d)$

$\Delta_{\gamma h_l}^{\mathbf{k}_l} f(\mathbb{x}) = \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\sum_{j=0}^{\mathbf{k}_l}(-1)^j\binom{\mathbf{k}_l}{j} f(x_1,\dots,x_l+(\mathbf{k}_l-j)h_l+(\mathbf{k}_l-j_{k_1})h_l+\dots+(\mathbf{k}_l-j_{k_l})h_l,\dots,x_d)$ 10

By induction on $\mathbf{k}_l$, can be proved the above definition. If $\mathbf{k}_l = 1 \rightarrow$

$\Delta_{\gamma h_l}^{1} f(\mathbb{x}) = \Delta_{\gamma h_l}^{1} f(x_1,\dots,x_l,\dots,x_d) = f(x_1,\dots,x_l+\gamma h_l,\dots,x_d) - f(x_1,\dots,x_l,\dots,x_d)$

$= \sum_{j=0}^{\gamma-1}[f(x_1,\dots,x_l+jh_l+h_l,\dots,x_d) - f(x_1,\dots,x_l+jh_l,\dots,x_d)]$

$\Delta_{\gamma h_l}^{1} f(\mathbb{x}) = \sum_{j=0}^{\gamma-1}\Delta_{h_l} f(x_1,\dots,x_l+jh_l,\dots,x_d)$

For arbitrary $\mathbf{k_l}\in\mathbb{N}$ and by Lemma 3(b)$\rightarrow$

$\Delta_{\gamma h_l}^{\mathbf{k}_l+1} f(\mathbb{x}) = \Delta_{\gamma h_l}^{\mathbf{k}_l}[\Delta_{\gamma h_l}^{1} f(\mathbb{x})]$

$= \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\Delta_{h_l}^{\mathbf{k}_l}[\Delta_{\gamma h_l}^{1}(f(x_1,\dots,x_l+(\mathbf{k}_l-j_{k_1})h_l+(\mathbf{k}_l-j_{k_2})h_l+\dots+(\mathbf{k}_l-j_{k_l})h_l,\dots,x_d))]$

$= \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\Delta_{h_l}^{\mathbf{k}_l}\left[\sum_{j_{k_l+1}=0}^{\gamma-1}\Delta_{h_l}\left(f(x_1,\dots,x_l+(\mathbf{k}_l-j_{k_1})h_l+(\mathbf{k}_l-j_{k_2})h_l+\dots+(\mathbf{k}_l-j_{k_l})h_l+(\mathbf{k}_l-j_{k_l+1})h_l,\dots,x_d)\right)\right]$

$= \sum_{j_{k_1}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\sum_{j_{k_l+1}=0}^{\gamma-1}\Delta_{h_l}^{\mathbf{k}_l+1} f(x_1,\dots,x_l+(\mathbf{k}_l-j_{k_1})h_l+(\mathbf{k}_l-j_{k_2})h_l+\dots+(\mathbf{k}_l-j_{k_l})h_l+(\mathbf{k}_l-j_{k_l+1})h_l,\dots,x_d)$

Form Eq 10 $\rightarrow$ for h_l, $|h_l|\le\delta$

$\left|\Delta_{\gamma h_l}^{\mathbf{k}_l} f(\mathbb{x})\right| \le \sum_{j_{k_2}=0}^{\gamma-1}\dots\sum_{j_{k_m}=0}^{\gamma-1}\dots\sum_{j_{k_l}=0}^{\gamma-1}\left\|\Delta_{h_l}^{\mathbf{k}_l} f(x_1,\dots,x_l+(\mathbf{k}_l-j_{k_2})h_l+(\mathbf{k}_l-j_{k_3})h_l+\dots+(\mathbf{k}_l-j_{k_l})h_l,\dots,x_d)\right\|_p$

$\le \gamma^{\mathbf{k}_l}\omega_{\mathbf{k}_l}(f,\delta)_p$.

Conclusion

This paper studies a new mixed difference for $\Bbbk$ -vectors in d-variables. Thus, a new modulus of $\Bbbk$ -mixed smoothness is obtained, and the fundamental modulus characteristics of $\Bbbk$ -mixed smoothness are also proved. Also, it can be concluded that the $\Bbbk$ -mixed modulus of smoothness achieves most of the inequalities achieved by the usual modulus of smoothness, such as monotonicity and linearity, in addition to a basic inequality, which is the boundness of functions belonging to the space L_p, $0 < p < 1$, with a mixed difference of order $\Bbbk$.

Acknowledgment

Thank God and everyone who supported us to accomplish this work.

Authors' Declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Babylon.

Authors' Contribution Statement

This work was carried out in collaboration between all authors. E. S. B, wrote and edited the manuscript with revisions idea. R. A. K, analysis the data with manuscript design. All authors read and approved the final manuscript.

References

1. Kasim NM. New Theorems in Approximation Theory. Baghdad Sci J. 2010; 7(2): 1056-1060. https://doi.org/10.21123/bsj.2010.7.2.1056-1060.
2. Aal-Rkhais HA, Kamil AH, Al Oweidi KF. The Approximation of Weighted Hölder Functions by Fourier-Jacobi Polynomials to the Singular Sturm-Liouville Operator. Baghdad Sci J. 2022; 19(6): 1387-1392 https://dx.doi.org/10.21123/bsj.2022.6128
3. Nikolsky SM. Functions with Dominating Mixed Derivatives Satisfying Multiple Holder Conditions. Sib Math J. 1963; 4(6): 1342-1364.
4. Najafov AM, Gasimova AM. On Some Differential Properties of Functions in Lizorkin-Triebel-Morrey Spaces with Dominant Mixed Derivatives. Adv Appl Math Sci. 2021; 20(9): 1909 -1921
5. Akgün R. Fractional Order Mixed Difference Operator and Its Applications in Angular Approximation. Hacet J Math Stat. 2020; 49(5): 1594 – 1610 https://doi.org/10.15672/hujms.569410
6. Zhao J, Du W-S, Chen Y. New Generalizations and Results in Shift-Invariant Subspaces of Mixed-Norm Lebesgue Spaces $Lp(R^d)$. Mathematics. 2021; 9(3): 1-10. https://doi.org/10.3390/math9030227.
7. Zhao J, Kostic M, Du W-S. On New Decomposition Theorems for Mixed-Norm Besove Spaces with Ingredient Modulus of Smoothness. Symmetry. 2023; 15(3): 1-17. https://doi.org/10.3390/sym15030642.
8. Potapov MK, Simonov BV. Estimates of Partial Moduli of Smoothness in Metrics of $Lp_{1\infty}$ and $L_{\infty p2}$ Through Partial Moduli of Smoothness in Metrics of Lp_1p_2. Moscow Univ Math Bull. 2020; 75: 1–15. https://doi.org/10.3103/S0027132220010015.
9. Potapov MK, Simonov BV. Refinement of Relations between Mixed Moduli of Smoothness in the Metrics of Lp and L∞. Math Notes. 2021; 110: 368–385. https://doi.org/10.1134/S0001434621090054.
10. Zygmund A. Trigonometric series. 3rd Ed . Volume I & II. United Kingdom: Cambridge University Press;1958728.p https://doi.org/10.1017/CBO9781316036587 .
11. Castillo RE, Rafeiro H. An Introductory Course in Lebesgue Spaces. 1st edition. Switzerland: Springer; 2016. 461p. https://doi.org/10.1007/978-3-319-30034-4.

Published Online First: July, 2024
https://doi.org/10.21123/bsj.2024.9726
P-ISSN: 2078-8665 - E-ISSN: 2411-7986
Baghdad Science Journal

مقياس النعومة $\mathbb{k}$ - المختلط

رحاب عامر كامل[1]، ايمان سمير بهيه[2،3]

[1]قسم الرياضيات، كلية التربية للعلوم الصرفة ،جامعة بابل، الحلة، العراق.
[2]قسم الرياضيات, كلية التربية، جامعة الزهراء للبنات، كربلاء، العراق.
[3]قسم الرياضيات، كلية التربية للعلوم الصرفة ،جامعة بابل، الحلة، العراق.

الخلاصة

يعتبر معامل او مقياس النعومة من المواضيع الاساسية والمهمة في التحليل الرياضي وتطبيقاته المتعددة. حيث يستخدم قياس نعومة الدوال معامل النعومة لاستنتاج الخصائص التقريبية للدوال التي يتعذر ايجاد قيمة دقيقة لها. لمقياس النعومة دور اساسي في العديد من مجالات الرياضيات وله تطبيقات مختلفة في نظرية التقريب و في التحليل العددي وخاصة في تحليل الطرق العددية لحل المعادلات التفاضلية ومشاكل التحسين والمعادلات التكاملية و ومعدل التقارب . تناول العديد من الباحثين في دراستهم معامل النعومة لدالة ذات متغير واحد ولكن قلة من الباحثين الذين تناولوا مقياس النعومة لدالة ذات عدة متغيرات. يهدف هذا البحث الى ملئ هذه الفجوة حيث عرف مقياس نعومة مختلط جديد لدالة تنتمي لفضاء الدوال القابلة للتكامل لبيكيا ذات d من المتغيرات. كما اثبتت العديد من الخصائص الاساسية لمقياس النعومة ولكن باستخدام مقياس نعومة مختلط لدالة ذات فرق منتهي مختلط ولـ d من المتغيرات كالخاصية الخطية والرتابة اضافة الى ذلك دراسة خاصية اساسية وهي المحدودية لدوال تنتمي لفضاء الدوال القابلة للتكامل والقياس.

الكلمات المفتاحية: المشتقات المختلطة، الفرق المنتهي المختلط، معامل النعومة المختلط، الخاصية الرتيبة، فضاء الدوال القابلة للقياس.