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Outcome for the case (11,10; 1)

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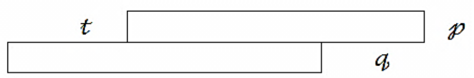
Abstract

Through this work R is a commutative ring with 1, $\mathcal{F}$ is a free R-module and $\mathcal{D}_i$ is the divided power algebra of degree i. A partition of length $\ell(\lambda)$ = n is a sequence $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. The weight of a partition λ is $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. A relative sequence is a pair (λ, μ) of sequences such that $\mu \leq \lambda$ and denoted by ${}^{\lambda}/_{\mu}$. If both (λ) and μ are partitions then the relative sequence $\left({}^{\lambda}/_{\mu}\right)$ is called a skew partition. Let $\mathcal{Z}_{21}$ be the free generator of divided power algebra $\mathcal{D}(\mathcal{Z}_{21})$ in one generator. The divided power element $\mathcal{Z}_{21}^{(e)}$ of degree e of the free generator $\mathcal{Z}_{21}$ acts on $\mathcal{D}_{p+e} + \mathcal{D}_{q-e}$ by place polarization of degree e from place 1 to place 2. The graded algebra $\mathcal{A} = \mathcal{D}(\mathcal{Z}_{21})$ acts on the graded module $\mathcal{M} = \mathcal{D}_{p+e} + \mathcal{D}_{q-e} = \sum \mathcal{M}_{q-e}$, the degree of the second factor determines the grading. M is a graded left $A = -$odule, where for w $= \mathcal{Z}_{21}^{(e)} \in \mathcal{A}$ and $v \in \mathcal{D}_{p+e} + \mathcal{D}_{q-e}$, so we have: $w(v) = \mathcal{Z}_{21}^{(e)} = (v) = \partial_{21}^{(e)}$. This article surveys the exactness of the Weyl resolution of the sequence for the case (11,10; 1) after getting the contracting homotopy of $\{\mathcal{S}_i\}$; where i = 0,1, …, 8, and the terms of the characteristic-free resolution for it.

Keywords: Divided power algebra, Letter place, Overlap, Place polarizations, Resolution, Weyl module.

Introduction

Writer in [1] discuss $\mathcal{K}_{\lambda/\mu}\mathcal{F}$ as:

$\lambda/\mu =$ 

For $\mathcal{K}_{\lambda/\mu}\mathcal{F} = \mathrm{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu}: \mathcal{DF} \longrightarrow \Lambda\mathcal{F}$ (Weyl map), so

$$\sum \mathcal{D}_{p+k}\otimes\mathcal{D}_{q-k} \xrightarrow{\square} \mathcal{D}_p\otimes\mathcal{D}_q \xrightarrow{d'_{\lambda/\mu}} \mathcal{K}_{\lambda/\mu} \longrightarrow 0$$

Searchers in[2] define the representation of the group. For case (11,10;1) the authors treatise this treatise as an application of the Weyl resolution in cases (4,4,3) [3], this can be employed for the differential subordination and superordination, fuzzy maximal sub-modules, almost projective semimodules and semigroup ideals and right n-derivation in 3-prime near-rings [4-7]. Authors in [8,9] study the hyper fuzzy AT-ideals of AT-algebra and spectrum of secondary submodules which can be also apply to this idea.

Main Upshots

As in [1, 5] for the case (11,10;1) gain

$\mathcal{M}_0 = \mathcal{D}_{10}\otimes\mathcal{D}_{10}$

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$\mathcal{M}_1 = \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{10+b} \otimes \mathcal{D}_{10-b}$; where $2 \leq b \leq 10$

$\mathcal{M}_2 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$; where $3 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_3 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$; where $4 \leq |b| = \sum_{i=1}^{3} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_4 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$; where $5 \leq |b| = \sum_{i=1}^{4} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_5 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$;
where $6 \leq |b| = \sum_{i=1}^{5} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_6 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$;

where $7 \leq |b| = \sum_{i=1}^{6} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_7 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{Z}_{21}^{(b_7)} x \mathcal{D}_{10+|b| \otimes \mathcal{D}_{10-|b|}}$;

where $8 \leq |b| = \sum_{i=1}^{7} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_8 = \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{Z}_{21}^{(b_7)} x \mathcal{Z}_{21}^{(b_8)} x \mathcal{D}_{10+|b|} \otimes \mathcal{D}_{10-|b|}$;

where $9 \leq |b| = \sum_{i=1}^{8} b_i \leq 10$ and $b_1 \geq 2$

$\mathcal{M}_9 = \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{D}_{20} \otimes \mathcal{D}_0$

Explore the Upshots

The construction and the exactness of the sequence of the contracting homotopies $\{\mathcal{S}_i\}$ where i =1,2,…,8 discuss in this section as follows

$$S_0: \mathcal{M}_0 \to \mathcal{M}_1 \ , S_0\left(\left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(10)}2^{(K)} \\ 2^{(10-K)} \end{smallmatrix}\right)\right) = \begin{cases} \mathcal{Z}_{21}^{(K)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(10+K)} \\ 2^{(10-K)} \end{smallmatrix}\right) & ; \text{if } K = 2,3,4,\ldots.9 \\ 0 & ; \text{if } K \leq 1 \end{cases} ,$$

$$S_1: \mathcal{M}_1 \to \mathcal{M}_2 \ , S_1\left(\mathcal{Z}_{21}^{(K+1)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+K)}2^{(v)} \\ 2^{(9-K-v)} \end{smallmatrix}\right)\right) = \begin{cases} \mathcal{Z}_{21}^{(K+1)} \chi \mathcal{Z}_{21}^{(v)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+K+v)} \\ 2^{(9-K-v)} \end{smallmatrix}\right) & ; \text{if } v = 1,2,\ldots,8 \\ 0 & ; \text{if } v = 0 \end{cases} ,$$

$S_2: \mathcal{M}_2 \to \mathcal{M}_3$

$$S_2\left(\mathcal{Z}_{21}^{(K_1+1)} \chi \mathcal{Z}_{21}^{(K_2)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$
$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)} \chi \mathcal{Z}_{21}^{(K_2)} \chi \mathcal{Z}_{21}^{(v)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ; \text{if } v = 1,2,\ldots,7 \\ 0 & ; \text{if } v = 0 \end{cases} \quad ; \text{where } |K| = K_1 + K_2,$$

$S_3: \mathcal{M}_3 \to \mathcal{M}_4$

$$S_3\left(\mathcal{Z}_{21}^{(K_1+1)} \chi \mathcal{Z}_{21}^{(K_2)} \chi \mathcal{Z}_{21}^{(K_3)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$
$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)} \chi \mathcal{Z}_{21}^{(K_2)} \chi \mathcal{Z}_{21}^{(K_3)} \chi \mathcal{Z}_{21}^{(v)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ; \text{if } v = 1,2,\ldots,6 \\ 0 & ; \text{if } v = 0 \end{cases} \quad ; \text{where } |K| = K_1 + K_2 + K_3.$$

$S_4: \mathcal{M}_4 \to \mathcal{M}_5$.

$$S_4\left(\mathcal{Z}_{21}^{(K_1+1)} \chi \mathcal{Z}_{21}^{(K_2)} \chi \mathcal{Z}_{21}^{(K_3)} \chi \mathcal{Z}_{21}^{(K_4)} \chi \left(\begin{smallmatrix} W \\ W' \end{smallmatrix} \middle| \begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

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$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ;\ if\ v = 1,2,\ldots,5 \\ 0 & ;if\ \ v=0 \end{cases};$$

where $|K| = K_1 + K_2 + K_3 + K_4$.

$S_5 : \mathcal{M}_5 \to \mathcal{M}_6$.

$$S_5\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ;if\ v = 1,2,\ldots,4 \\ 0 & ;if\ \ v=0 \end{cases};$$

where $|K| = K_1 + K_2 + K_3 + K_4 + K_5$.

$S_6 : \mathcal{M}_6 \to \mathcal{M}_7$.

$$S_6\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ;if\ v = 1,2,3 \\ 0 & ;if\ \ v=0 \end{cases};$$

where $|K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6$.

$S_7 : \mathcal{M}_7 \to \mathcal{M}_8$

$$S_7\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\mathcal{Z}_{21}^{(K_7)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\mathcal{Z}_{21}^{(K_7)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ;if\ v = 1,2 \\ 0 & ;if\ \ v=0 \end{cases};$$

where $|K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7$.

$S_8 : \mathcal{M}_8 \to \mathcal{M}_9$

$$S_8\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\mathcal{Z}_{21}^{(K_7)}\chi\mathcal{Z}_{21}^{(K_8)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$= \begin{cases} \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(K_5)}\chi\mathcal{Z}_{21}^{(K_6)}\chi\mathcal{Z}_{21}^{(K_7)}\chi\mathcal{Z}_{21}^{(K_8)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) & ;if\ v = 1 \\ 0 & ;if\ \ v=0 \end{cases};$$

where $|K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8$.

Now to prove that $S_i\partial_x + \partial_x S_{i+1} = id_{\mathcal{M}_{i+1}}$, for i = 0,1,…,8.

$$S_0\partial_x\left(\mathcal{Z}_{21}^{(K+1)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K)}2^{(v)} \\ 2^{(9-K)} \end{smallmatrix}\right)\right) = S_0\partial_{21}^{(K+1)}\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K)}2^{(v)} \\ 2^{(9-K)} \end{smallmatrix}\right)$$

$$= \binom{K+1+v}{v}\mathcal{Z}_{21}^{(K+1+v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K+v)} \\ 2^{(9-K-v)} \end{smallmatrix}\right)$$

And

$$\partial_x S_1\left(\mathcal{Z}_{21}^{(K+1)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K)}2^{(v)} \\ 2^{(9-K)} \end{smallmatrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(K+1)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K+v)} \\ 2^{(9-K-v)} \end{smallmatrix}\right)\right)$$

$$= -\binom{K+1+v}{v}\mathcal{Z}_{21}^{(K+1+v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K+v)} \\ 2^{(9-K-v)} \end{smallmatrix}\right) + \mathcal{Z}_{21}^{(K+1)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+K)}2^{(v)} \\ 2^{(9-K)} \end{smallmatrix}\right)$$

Clearly $S_0\partial_x + \partial_x S_1 = id_{\mathcal{M}_1}$.

$$S_1\partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$= S_1\left(-\binom{|K|+1}{K_2}\mathcal{Z}_{21}^{(|K|+1)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) + \mathcal{Z}_{21}^{(K_1+1)}\chi\partial_{21}^{(K_2)}\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$=\left(-\binom{|K|+1}{K_2}\mathcal{Z}_{21}^{(|K|+1)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) + \binom{K_2+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

And

$$\partial_x S_2\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)\right)$$

$$=\binom{|K|+1}{K_2}\mathcal{Z}_{21}^{(|K|+1)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right) - \binom{K_2+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+v)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right)$$

$$+\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\middle|\begin{smallmatrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{smallmatrix}\right);$$ where $|K| = K_1 + K_2$.

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Clearly $\quad S_1\partial_x + \partial_x S_2 = id_{\mathcal{M}_2}$

$$S_2\partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= S_2\left[\binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right) - \binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right) + \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\partial_{21}^{(K_3)}\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right]$$

$$= \binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right) - \binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right) + \binom{K_3+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right),$$

And

$$\partial_x S_3\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right) = \partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= -\binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_3+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right) + \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\partial_{21}^{(v)}\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$= -\binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_3+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right);\ \text{where }\ |K| = K_1 + K_2 + K_3.$$

Clearly $\quad S_2\partial_x + \partial_x S_3 = id_{\mathcal{M}_3}$.

$$S_3\partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= S_3\left[-\binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right) + \binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right) - \binom{K_3+K_4}{K_4}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3+K_4)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right) + \mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\partial_{21}^{(K_4)}\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right]$$

$$= -\binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_3+K_4}{K_4}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3+K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_4+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right),$$

And

$$\partial_x S_4\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$
$$= \partial_x\left(\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$
$$= \binom{(K_1+1+K_2)}{K_2}\mathcal{Z}_{21}^{(K_1+1+K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_2+K_3}{K_3}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2+K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_3+K_4}{K_4}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3+K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_4+v}{v}\mathcal{Z}_{21}^{(K_1+1)}\chi\mathcal{Z}_{21}^{(K_2)}\chi\mathcal{Z}_{21}^{(K_3)}\chi\mathcal{Z}_{21}^{(K_4)}\chi\mathcal{Z}_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

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$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\partial_{21}^{(v)}\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$=\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_4+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$; where $|K| = K_1 + K_2 + K_3 + K_4$

Clearly $S_3\partial_x + \partial_x S_4 = id_{M_4}$.

$S_4\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)\right)$

$= S_4\Bigg[\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)-$

$\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)+$

$\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)-$

$\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)+$

$Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi\partial_{21}^{(K_5)}\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)\Bigg]$

$=\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_5+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

And

$\partial_x S_5\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)}2^{(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)\right)$

$=\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|)+(v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)\right)$

$=-\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_5+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\partial_{21}^{(v)}\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$=-\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$+\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

$-\binom{K_5+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+v)}\chi\left(\begin{matrix} W \\ W' \end{matrix}\middle|\begin{matrix} 1^{(11+|K|+v)} \\ 2^{(9-|K|-v)} \end{matrix}\right)$

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$$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right);$$

$where\ |K| = K_1 + K_2 + K_3 + K_4 + K_5.$

Clearly $S_4\partial_x + \partial_x S_5 = id_{\mathcal{M}_5}$.

$$S_5\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= S_5\Bigg[-\binom{K_1+1+K_2}{K_2}Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$$

$$\binom{K_2+K_3}{K_3}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)-$$

$$\binom{K_3+K_4}{K_4}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$$

$$\binom{K_4+K_5}{K_5}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)-$$

$$\binom{K_5+K_6}{K_6}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$$

$$Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi\partial_{21}^{(K_5)}Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\Bigg]$$

$$= -\binom{K_1+1+K_2}{K_2}Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_2+K_3}{K_3}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_3+K_4}{K_4}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_4+K_5}{K_5}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_5+K_6}{K_6}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_6+v}{v}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

And

$$\partial_x S_6\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= \partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)+(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$

$$= \binom{K_1+1+K_2}{K_2}Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_2+K_3}{K_3}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_3+K_4}{K_4}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_4+K_5}{K_5}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_5+K_6}{K_6}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_6+v}{v}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\partial_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$= \binom{K_1+1+K_2}{K_2}Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_2+K_3}{K_3}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_3+K_4}{K_4}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_4+K_5}{K_5}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+\binom{K_5+K_6}{K_6}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$-\binom{K_6+v}{v}Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

$$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right);$$

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where $|K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6$.
Clearly $S_5\partial_x + \partial_x S_6 = id_{\mathcal{M}_6}$.

$$S_6\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$
$$= S_6\Bigg[\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi\partial_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\Bigg]$$
$$=\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_7+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

And

$$\partial_x S_7\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$
$$=\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)+(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$$
$$=-\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_7+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\partial_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$=-\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$-\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$
$$+\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$$

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$-\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_7+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right);$
where $|K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7$.
Clearly $S_6\partial_x + \partial_x S_7 = id_{\mathcal{M}_7}$.

$S_7\partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$
$= S_6\left[-\binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right.$
$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$
$\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)-$
$\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$
$\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)-$
$\binom{K_7+K_8}{K_8} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)+$
$\left.Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi \partial_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right]$
$= -\binom{K_1+1+K_{,2}}{K_2} Z_{21}^{(K_1+1+K_{,2})}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_{,3}+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_{,5}+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_7+K_8}{K_{87}} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_8+v}{v} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8+v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
And
$\partial_x S_8\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)}2^{(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$
$= \partial_x\left(Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|)+(v)}\\2^{(9-|K|-v)}\end{matrix}\right)\right)$
$= \binom{K_1+1+K_2}{K_2} Z_{21}^{(K_1+1+K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2+K_{,3})}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3+K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4+K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K_,|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5+K_6)}\chi Z_{21}^{(K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$-\binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6+K_7)}\chi Z_{21}^{(K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$
$+\binom{K_7+K_8}{K_8} Z_{21}^{(K_1+1)}\chi Z_{21}^{(K_2)}\chi Z_{21}^{(K_3)}\chi Z_{21}^{(K_4)}\chi Z_{21}^{(K_5)}\chi Z_{21}^{(K_6)}\chi Z_{21}^{(K_7+K_8)}\chi Z_{21}^{(v)}\chi\left(\begin{matrix}W\\W'\end{matrix}\middle|\begin{matrix}1^{(11+|K|+v)}\\2^{(9-|K|-v)}\end{matrix}\right)$

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$$-\binom{K_8+v}{v} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8+v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$+Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi \partial_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$= \binom{(K_1+1+K_2)}{K_2} Z_{21}^{(K_1+1+K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$- \binom{K_2+K_3}{K_3} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2+K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$+ \binom{K_3+K_4}{K_4} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3+K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$- \binom{K_4+K_5}{K_5} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4+K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$+ \binom{K_5+K_6}{K_6} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5+K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$- \binom{K_6+K_7}{K_7} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6+K_7)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$+ \binom{K_7+K_8}{K_8} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_8)} \chi Z_{21}^{(K_7+K_8)} \chi Z_{21}^{(v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$-\binom{K_8+v}{v} Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(8+v)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|+v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right)$$
$$+Z_{21}^{(K_1+1)} \chi Z_{21}^{(K_2)} \chi Z_{21}^{(K_3)} \chi Z_{21}^{(K_4)} \chi Z_{21}^{(K_5)} \chi Z_{21}^{(K_6)} \chi Z_{21}^{(K_7)} \chi Z_{21}^{(K_8)} \chi \left(\begin{array}{c|c} W & 1^{(11+|K|)} 2^{(v)} \\ W' & 2^{(9-|K|-v)} \end{array}\right);$$

$where\ |K| = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8.$
Clearly $S_7 \partial_x + \partial_x S_8 = id_{\mathcal{M}_8}$
Thus $\{\mathcal{S}_i\}$; where i = 0, 1, 2, …, 8 is a contracting homotopy[10] which implies the complex

$$0 \longrightarrow \mathcal{M}_9 \longrightarrow \mathcal{M}_8 \longrightarrow \mathcal{M}_7 \longrightarrow \mathcal{M}_6 \longrightarrow \mathcal{M}_5 \longrightarrow \mathcal{M}_4 \longrightarrow \mathcal{M}_3 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0$$

is exact.

Results and Discussion

In this article, the terms of the characteristic-free resolution for the situation (11,10; 1) and the exactness of the Weyl resolution of the sequence are surveyed after obtaining the contracting homotopy of $\{\mathcal{S}_i\}$; where i = 0,1, …, 8. To get the results of this work the important computations which are $S_{i-1}\partial_x + \partial_x S_i = id_{\mathcal{M}_i}$; where $i = 0,1,2,\dots,8$ proved.

Conclusion

The exact complex sequence of the terms of the characteristic-free resolution got in this work from the contracting homotopies of $\{\mathcal{S}_i\}$ where $i = 0,1,2,\dots,8$ for the case (11,10;1) after finding all the terms of the Weyl module and the Weyl module resolution, also by using the form of the letter place the maps of $\{\mathcal{S}_i\}$ where $i = 0,1,2,\dots,8$ are described and $\partial_{21}^{(e)}$ the composition of place polarizations from places 1 and 2 of degree e to the negative place {1′, 2′,…, 12′} act on $.\mathcal{D}_{10+e} \oplus \mathcal{D}_{10-e}$.

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Authors’ Declaration

- Conflicts of Interest: None

- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at University of Baghdad.

Authors' Contribution Statement

N. S. J. presented the idea and discussed it theoretically and checked the computations. W. K. J. made the computations for this work, N. S.J. and W. K. J. discussed and validates the results obtained by manual calculations.

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نتيجة للحالة (11,10; 1)

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الخلاصة

خلال هذا العمل R هي حلقة ابدالية ذات 1، $\mathcal{F}$ هو مقاس R الحر و $\mathcal{D}_i$ هو جبر قوى القسمة من الدرجة i . التجزئة ذات الطول $\ell(\lambda) = n$ هي السلسلة $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ من الصحيحات غير السالبة وبترتيب غير متزايد $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. وزن التجزئة λ هو $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. المتسلسلة النسبية هي الزوج (λ, μ) من المتسلسلات بحيث ان $\mu \leq \lambda$ ويرمز لها بالرمز λ/μ. اذا كان كلا من λ و μ تجزئة فان السلسلة النسبية λ/μ تسمى تجزئة منحرفة. ليكن $\mathcal{Z}_{21}$ المولد الحر لجبر قوى القسمة $\mathcal{D}(\mathcal{Z}_{21})$ بمولد واحد. عنصر قوى القسمة $\mathcal{Z}_{21}^{(e)}$ من الدرجة e للمولد الحر $\mathcal{Z}_{21}$ يمثل في $\mathcal{D}_{p+e} + \mathcal{D}_{q-e}$ بالاستقطاب المكاني من الدرجة e من المكان 1 الى المكان 2. التدرج الجبري $\mathcal{A} = \mathcal{D}(\mathcal{Z}_{21})$ يمثل بتدرج المقاس $\mathcal{M} = \mathcal{D}_{p+e} + \mathcal{D}_{q-e} = \sum \mathcal{M}_{q-e}$، درجة المقسوم الثاني تعتبر التدرج.M هو مقاس A= الايسر المتدرج، حيث لكل $w = \mathcal{Z}_{21}^{(e)} \in \mathcal{A}$ و $\mathscr{v} \in \mathcal{D}_{p+e} + \mathcal{D}_{q-e}$، فانه لدينا $w(\mathscr{v}) = \mathcal{Z}_{21}^{(e)} = (\mathscr{v}) = \partial_{21}^{(e)}$. البحث يدرس تمام تحلل وايل للمتسلسلة للحالة (11,10; 1) بعد الحصول على التعاقد الهوموتوبي لـ $\{\mathcal{S}_i\}$ ؛ حيث $i = 0,1,2, \ldots, 8$ ، و عناصر تحلل المميز - الحر لها.

الكلمات المفتاحية: جبر قوى القسمة، موقع الحرف، تداخل، موقع الاستقطابات، تحلل، مقاس وايل.