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New Subclasses of Bi-Univalent Functions Associated with Exponential Functions and Fibonacci Numbers

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Abstract

Lewin discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient, Sakar and Wanas defined two new subclasses of bi-univalent functions and obtained upper bounds for the elementary coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses, Dziok et al. introduced the class SLM_{α} of α-convex shell-like functions, and they indicated a useful connection between the function $\tilde{p}(z)$ and Fibonacci numbers. Recently, many bi-univalent function classes, based on well-known operators like Sãlãgean operator, Tremblay operator, Komatu integral operator, Convolution operator, Al-Oboudi Differential operator and other, have been defined. The aims of this paper is to introduce two new subclasses of bi-univalent functions using the subordination and the Komatu integral operator which are involved the exponential functions and shell-like curves with Fibonacci numbers, also find an estimate of the initial coefficients for these subclasses. The first subclass was defined using the subordination of the shell-like curve functions related to Fibonacci numbers and the second subclass was defined using the subordination of the exponential function. The Komatu integral operator was used in each of these subclasses. Limits were obtained for the elementary coefficients, specifically the second and third coefficients for these subclasses.

Keywords: Bi-univalent functions, Coefficient bounds, Exponential function, Fibonacci numbers, Komatu integral operator, Subordination.

Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{U} = \{z: z \in \mathbb{C} \; and \; |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Further, let $\mathcal{A}$ be the class of functions analytic in $\mathbb{U}$, thus satisfying the condition

$$f(0) = f'(0) - 1 = 0.$$

In addition, each of the functions f in $\mathcal{A}$ has the following Taylor series expansion

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n. \quad 1$$

Suppose S is a subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

By the Koebe One-Quarter Theorem [1], it is known that the range of every function in $\mathcal{S}$ contains the disk

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$\{w : |w| < 1/4\}$. Therefore, every univalent function f has an inverse f^{-1}, so that:

$$f(f(z)) = z\ (z \in \mathbb{U})\ and\ f(f^{-1}(w)) = w\ (\ |w| < r_0(f); r_0(f) \geq 1/4)$$

$$g(w) = f^{-1}(w) = w + b_2w^2 + b_3w^3 + \cdots$$

$$= w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse $g = f^{-1}$ are univalent in $\mathbb{U}$. Let Σ be the class of all bi-univalent functions. Examples of functions in the class Σ are

$$\frac{z}{1-z}\quad,\quad -\log(1-z)\ ,\qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Let us denote by B the class of bounded or Schwarz functions $w(z)$ which are analytic in the open unit disc $\mathbb{U}$ and satisfying:

$$w(z) = \sum_{n=1}^{\infty} c_n z^n\ ,\qquad w(0) = 0, |w(z)| < 1.$$

Definition 1: [1] Consider two functions f and g analytic in $\mathbb{U}$, it is said that f is subordinate to g (symbolically $f \prec g$) if there exists a bounded function $w(z) \in B$ for which $f(z) = g\big(w(z)\big)$. This definition is known as the principle of subordination.

Lemma 1: [1] For two analytic functions $u(z), b(w)$, where $(u(0) = b(0) = 0, |u(z)| < 1, |b(w)| < 1)$ suppose that:

$$u(z) = \sum_{n=1}^{\infty} x_n z^n\ (z \in \mathbb{U}),$$

$$b(w) = \sum_{n=1}^{\infty} y_n w^n\ (\ w \in \mathbb{U}).$$

Then:

$$|x_1| \leq 1, |x_2| \leq 1 - |x_1|^2, |y_1| \leq 1, |y_2| \leq 1 - |y_1|^2.$$

Lewin in[2] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient, Sakar and Wanas in[3] defined two new subclasses of bi-univalent functions and obtained upper bounds for the elementary coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses. The issue of estimating the first and second coefficients in subclasses of bi-univalent functions is still the focus of attention of many researchers in this field, as shown in [4-6]. It is known that the exponential function has an expansion in the Taylor series as follows

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Shi et al. in[7] obtained Hankel determinant of third order bounds for univalent functions subclasses S_e^* and C_e which are associated with exponential functions. The subclasses S_e^* and C_e are defined as follows:

$$S_e^* = \left\{f \in S :\ \frac{zf'(z)}{f(z)} \prec e^z\ , z \in \mathbb{U}\right\}$$

$$C_e = \left\{f \in S :\ \frac{\big(zf'(z)\big)'}{f'(z)} \prec e^z\ , z \in \mathbb{U}\right\}$$

In[8], Zaprawa improved the result in[7] for the bounds of Hankel determinant of third order in the subclass S_e^* . Besides that, Zaprawa[8] obtained the bounds of Hankel determinant of the third order in the subclass $\mathcal{K}_e$ of univalent functions associated with exponential functions that are defined as follows:

$$\mathcal{K}_e = \left\{f \in S :\ 1 + \frac{zf''(z)}{f'(z)} \prec e^z\ , z \in \mathbb{U}\right\}$$

In[9] Dziok et al. introduced the class SLM_{α} of α-convex shell-like functions, and they indicate a useful connection between the function $\tilde{p}(z)$ and Fibonacci numbers.

Lemma 2: [9] Let $\{u_n\}$ be the sequence of Fibonacci numbers:

$$\begin{cases} u_0 = 0, u_1 = 1 \\ u_{n+2} = u_{n+1} + u_n \quad ; (n \in \mathcal{N}_0) \end{cases}$$

then

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}\quad ,\quad \tau = \frac{1-\sqrt{5}}{2}$$

If one can set:

$$\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n = 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n$$

Then the coefficients $\tilde{p}_n$ satisfy:

$$\tilde{p}_n = \begin{cases} \tau & (n = 1) \\ 3\tau^2 & (n = 2) \\ \tau\tilde{p}_{n-1} + \tau^2\tilde{p}_{n-2} & (n = 3,4 \dots). \end{cases}$$

The function $\tilde{p}(z)$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$, For example, $\tilde{p}(0) = \tilde{p}\left(\frac{-1}{2\tau}\right) = 1$ and $\tilde{p}\left(e^{\pm i arccos\left(\frac{1}{4}\right)}\right) = \frac{\sqrt{5}}{5}$.

The expression $\tau^2 = \tau + 1$ can be used to obtain higher powers τ^n as a liner function of lower powers, recurrence relationships yield Fibonacci numbers u_n :

$$\tau^n = u_n\tau + u_{n-1} \quad ; \begin{cases} u_0 = 0, u_1 = 1 \\ u_{n+2} = u_{n+1} + u_n \end{cases}$$

(see [10] for details).

Definition 2: [10] The Komatu integral operator of $f \in \mathcal{A}$ is denoted by $\mathcal{K}_t^\eta f(z)$ and defined by:

$$\mathcal{K}_t^\eta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{t}{t+n-1}\right)^\eta a_n z^n \; ; \; (t > 0 \,, \eta \geq 0, z \in \mathbb{U})$$

$$= \frac{t^\eta}{\Gamma(\eta)} \int_0^1 \xi^{t-2} \left(\log\frac{1}{\xi}\right)^{\eta-1} f(z\xi) d\xi.$$

Recently, many bi-univalent function classes, based on well-known operators like the Sãlãgean operator, Tremblay operator, Komatu integral operator as in[10-12], Convolution operator, Al-Oboudi Differential operator and other as in[13-15], have been defined. To define our new classes, where the Komatu operator is used which is defined as follows:

Definition 3: A function $f \in \Sigma$ is said to be in the class:

$$\mathcal{M}_{\Sigma,\rho}^{\eta,t}(\tilde{\mathfrak{p}}) \; (1 > \rho \geq 0, t > 0, \eta \geq 0, z, w \in \mathbb{U})$$

If the following subordination relationships are satisfied:

$$\left[(1-\rho)\frac{\mathcal{K}_t^\eta f(z)}{f(z)} + \rho\left(1 + \frac{z\left(\mathcal{K}_t^\eta f(z)\right)'}{f'(z)}\right)\right] \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad 3$$

and

$$\left[(1-\rho)\frac{\mathcal{K}_t^\eta g(w)}{g(w)} + \rho\left(1 + \frac{z\left(\mathcal{K}_t^\eta g(w)\right)'}{g'(w)}\right)\right] \prec \tilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \qquad 4$$

where the function $g(w)$ is given by 2 and $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Definition 4: A function $f \in \Sigma$ is said to be in the class:

$$\mathcal{H}_{\Sigma,\rho}^{\eta,t}(e^z) \; (1 \geq \rho \geq 0, t > 0, \eta \geq 0, z, w \in \mathbb{U})$$

If the following subordination relationships are satisfied:

$$1 + \frac{1}{\tau}\left[\left(\mathcal{K}_t^\eta f(z)\right)' + \rho z\left(\mathcal{K}_t^\eta f(z)\right)'' - 1\right] \prec e^z \qquad 5$$

and

$$1 + \frac{1}{\tau}\left[\left(\mathcal{K}_t^\eta g(w)\right)' + \rho w\left(\mathcal{K}_t^\eta g(w)\right)'' - 1\right] \prec e^w, \qquad 6$$

where the function $g(w)$ is given by 2 and $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Results and Discussion

First, the bounds of the coefficients $|a_2|, |a_3|$ is obtained for functions in $\mathcal{M}_{\Sigma,\rho}^{\eta,t}(\tilde{\mathfrak{p}})$.

Theorem 1: If $f \in \mathcal{M}_{\Sigma,\rho}^{\eta,t}(\tilde{\mathfrak{p}})$, then

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$|a_2| \le$

$$min\left\{\frac{|\tau|}{(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}, \sqrt{\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}}\right\}$$

And

$$|a_3| \le \min\left\{\begin{array}{c}\frac{|\tau|\left(\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)|\tau| + (2\rho+(1-\rho)(1+3|\tau|))\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2\right)}{(1-\rho)^2\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2},\\ \frac{|\tau|(1+3|\tau|)\sqrt{1-\rho}\left(1-2\left(\frac{t}{t+2}\right)^{\eta}+\left(\frac{t}{t+1}\right)^{\eta}\right)+2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)\left[|\tau|(1+3|\tau|)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)\right]^{\frac{1}{2}}}{(1-\rho)^{\frac{3}{2}}\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}\end{array}\right\}$$

Proof: As $f \in \mathcal{M}_{\Sigma,\rho}^{\eta,t}(\tilde{\mathrm{p}})$, so by Definition 3 and using the principle of subordination in Definition 1, there exist Schwarz functions $u(z)$ and $b(w)$, one can write 3 and 4 as following

$$(1-\rho)\frac{\mathcal{K}_t^{\eta} f(z)}{f(z)} + \rho\left(1+\frac{z\left(\mathcal{K}_t^{\eta} f(z)\right)'}{f'(z)}\right) = \tilde{p}(u(z)) \quad 7$$

and

$$(1-\rho)\frac{\mathcal{K}_t^{\eta} g(w)}{g(w)} + \rho\left(1+\frac{z\left(\mathcal{K}_t^{\eta} g(w)\right)'}{g'(w)}\right) = \tilde{p}(b(w)), \quad 8$$

where $u(z) = x_1 z + x_2 z^2 + \cdots$ and $b(w) = y_1 w + y_2 w^2 + \cdots$ $(z, w \in \mathbb{U})$.

Using 1 and Definition 2 in 7 on expanding, it yields

$$(1-\rho)\frac{\mathcal{K}_t^{\eta} f(z)}{f(z)} + \rho\left(1+\frac{z\left(\mathcal{K}_t^{\eta} f(z)\right)'}{f'(z)}\right) = 1 + \left[\rho + (1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2\right]z$$
$$+\left[(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2^2 + (1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)a_3 + 2\rho\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2\right]z^2 + \cdots \quad 9$$

Using 2 and Definition 2 in 8 on expanding, it yields

$$(1-\rho)\frac{\mathcal{K}_t^{\eta} g(w)}{g(w)} + \rho\left(1+\frac{z\left(\mathcal{K}_t^{\eta} g(w)\right)'}{g'(w)}\right) = 1 + \left[\rho + (1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)b_2\right]w$$
$$+\left[(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)b_2^2 + (1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)b_3 + 2\rho\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)b_2\right]w^2 + \cdots$$

Since $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$, one can get

$$= 1 + \left[\rho + (1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2\right]w$$
$$+\left[(1-\rho)\left(2\left(\frac{t}{t+2}\right)^{\eta}-\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2^2 + (1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)a_3 + 2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2\right]w^2 + \cdots \quad 10$$

Again

$$\tilde{p}(u(z)) = 1 + \tilde{p}_1 x_1 z + (\tilde{p}_1 x_2 + \tilde{p}_2 x_1^2)z^2 + \cdots \quad 11$$

and

$$\tilde{p}(b(w)) = 1 + \tilde{p}_1 y_1 w + (\tilde{p}_1 y_2 + \tilde{p}_2 y_1^2)w^2 + \quad 12$$

Using (9) and (11) in (7) and equating the coefficients of z and z^2, lead to

$$\rho + (1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2 = \tilde{p}_1 x_1 \quad 13$$

and

$$(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2^2 + (1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)a_3 + 2\rho\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2 = \tilde{p}_1 x_2 + \tilde{p}_2 x_1^2. \quad 14$$

Using 10 and 12 in 8 and equating the coefficients of w and w^2, lead to

$$\rho + (1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2 = \tilde{p}_1 y_1 \quad 15$$

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and

$$(1-\rho)\left(2\left(\frac{t}{t+2}\right)^{\eta}-\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2^2+(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)a_3+2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)a_2=\tilde{p}_1y_2+\tilde{p}_2y_1^2. \quad 16$$

Subtracting 15 from 13, it yields

$$a_2=\frac{\tilde{p}_1(x_1-y_1)}{2(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)},$$

Applying triangle inequality, one can get

$$|a_2|\le\frac{|\tilde{p}_1|(|x_1|+|y_1|)}{2(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)},$$

Applying Lemma 1 and Lemma 2, one can obtain

$$|a_2|\le\frac{|\tau|}{(1-\rho)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}. \quad 17$$

Adding 14 and 16, it yields

$$a_2^2=\frac{\tilde{p}_1(x_2+y_2)+\tilde{p}_2(x_1^2+y_1^2)}{2(1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-\left(\frac{t}{t+1}\right)^{\eta}\right)},$$

Applying Lemma 1 and triangle inequality and taking the root of both sides, one can find

$$|a_2|\le\sqrt{\frac{|\tau|(2-|x_1|^2-|y_1|^2)+6\tau^2}{2(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}},$$

Again applying Lemma 1 and Lemma 2, one can obtain

$$|a_2|\le\sqrt{\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}}.$$

18

From 17 and 18, the first part of our theorem is fulfilled.

Subtracting 16 from 14, it is seen that

$$2(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)a_2^2+2(1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)a_3+4\rho\left(\left(\frac{t}{t+1}\right)^{\eta}-1\right)a_2=\tilde{p}_1(x_2-y_2)+\tilde{p}_2(x_1^2-y_1^2),$$

Therefore,

$$a_3=\frac{2(1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)a_2+4\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}{2(1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)}a_2+\frac{\tilde{p}_1(x_2-y_2)+\tilde{p}_2(x_1^2-y_1^2)}{2(1-\rho)\left(\left(\frac{t}{t+2}\right)^{\eta}-1\right)},$$

Applying triangle inequality and Lemma 1 and Lemma 2, one can obtain

$$|a_3|\le|a_2^2|+\frac{2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)}|a_2|+\frac{|\tilde{p}_1|(|x_2|+|y_2|)+|\tilde{p}_2|(|x_1^2|+|y_1^2|)}{2(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)},$$

$$|a_3|\le|a_2^2|+\frac{2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)}|a_2|+\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)}. \quad 19$$

From 17 and 19, one can get

$$|a_3|\le\frac{|\tau|^2}{(1-\rho)^2\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2}+\frac{2\rho|\tau|}{(1-\rho)^2\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)}+\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)},$$

$$|a_3|\le\frac{|\tau|^2}{(1-\rho)^2\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2}+\frac{|\tau|\left(2\rho+(1-\rho)(1+3|\tau|)\right)}{(1-\rho)^2\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)},$$

$$|a_3|\le\frac{|\tau|\left(\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)|\tau|+\left(2\rho+(1-\rho)(1+3|\tau|)\right)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2\right)}{(1-\rho)^2\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)^2}. \quad 20$$

Also from 18 and 19, one can get

$$|a_3|\le\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}+\frac{2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)}\sqrt{\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}}+\frac{|\tau|(1+3|\tau|)}{(1-\rho)\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)},$$

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$$|a_3| \leq \frac{|\tau|(1+3|\tau|)\sqrt{1-\rho}\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)+2\rho(|\tau|(1+3|\tau|))^{\frac{1}{2}}\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)^{\frac{1}{2}}+|\tau|(1+3|\tau|)\sqrt{1-\rho}\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}{(1-\rho)^{\frac{3}{2}}\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)},$$

$$|a_3| \leq \frac{|\tau|(1+3|\tau|)\sqrt{1-\rho}\left(1-2\left(\frac{t}{t+2}\right)^{\eta}+\left(\frac{t}{t+1}\right)^{\eta}\right)+2\rho\left(1-\left(\frac{t}{t+1}\right)^{\eta}\right)\left[|\tau|(1+3|\tau|)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)\right]^{\frac{1}{2}}}{(1-\rho)^{\frac{3}{2}}\left(1-\left(\frac{t}{t+2}\right)^{\eta}\right)\left(\left(\frac{t}{t+1}\right)^{\eta}-\left(\frac{t}{t+2}\right)^{\eta}\right)}. \quad 21$$

From 20 and 21, the second part of our theorem is fulfilled.

Theorem 2: If $f \in \mathcal{H}_{\Sigma,\rho}^{\eta,t}(e^z)$, then

$$|a_2| \leq min\left\{\frac{|\tau|}{2(1+\rho)\left(\frac{t}{t+1}\right)^{\eta}}, \sqrt{\frac{|\tau|}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{|\tau|}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}, \frac{|\tau|\left(|\tau|(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}+2(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right)}{4(1+2\rho)(1+\rho)^2\left(\frac{t}{t+2}\right)^{\eta}\left(\frac{t}{t+1}\right)^{2\eta}}\right\}.$$

Proof: As $f \in \mathcal{H}_{\Sigma,\rho}^{\eta,t}(e^z)$, then by Definition 4 and by using the principle of subordination in Definition 1, there exist Schwarz functions $u(z)$ and $b(w)$ such that

$u(z) = \sum_{n=1}^{\infty} x_n z^n$, $b(w) = \sum_{n=1}^{\infty} y_n w^n$, $(z \in \mathbb{U}, w \in \mathbb{U})$, thus one can write 5 and 6 as follows:

$$1+\frac{1}{\tau}\left[\left(\mathcal{K}_t^{\eta} f(z)\right)' + \rho z\left(\mathcal{K}_t^{\eta} f(z)\right)'' - 1\right] = e^{u(z)} \quad 22$$

and

$$1+\frac{1}{\tau}\left[\left(\mathcal{K}_t^{\eta} g(w)\right)' + \rho w\left(\mathcal{K}_t^{\eta} g(w)\right)'' - 1\right] = e^{b(w)}. \quad 23$$

On expanding, it yields

$$1+\frac{1}{\tau}\left[\left(\mathcal{K}_t^{\eta} f(z)\right)' + \rho z\left(\mathcal{K}_t^{\eta} f(z)\right)'' - 1\right] = 1+\frac{2}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 z + \frac{3}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_3 z^2 + \cdots \quad 24$$

and

$$1+\frac{1}{\tau}\left[\left(\mathcal{K}_t^{\eta} g(w)\right)' + \rho z\left(\mathcal{K}_t^{\eta} g(w)\right)'' - 1\right] = 1+\frac{2}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} b_2 w + \frac{3}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} b_3 w^2 + \cdots$$

Since $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$, one can get

$$1+\frac{1}{\tau}\left[\left(\mathcal{K}_t^{\eta} g(w)\right)' + \rho z\left(\mathcal{K}_t^{\eta} g(w)\right)'' - 1\right] = 1-\frac{2}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 w + \frac{3}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}(2a_2^2 - a_3)w^2 + \cdots \quad 25$$

Again

$$e^{u(z)} = 1 + x_1 z + \left(x_2 + \frac{x_1^2}{2}\right)z^2 + \cdots \quad 26$$

and

$$e^{u(z)} = 1 + y_1 w + \left(y_2 + \frac{y_1^2}{2}\right)w^2 + \cdots \quad 27$$

Using 24 and 26 in 22 and equating the coefficients of z and z^2, lead to

$$\frac{2}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 = x_1 \quad 28$$

and

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$$\frac{3}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_3 = x_2 + \frac{x_1^2}{2}.$$
29

Again using 25 and 27 in 23 and equating the coefficients of w and w^2, lead to

$$-\frac{2}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 = y_1$$
30

and

$$\frac{3}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}(2a_2^2 - a_3) = y_2 + \frac{y_1^2}{2}.$$
31

Adding 28 and 30, it yields

$$y_1 = -x_1$$

Thus

$$|y_1| = |x_1|. \tag{32}$$

Subtracting 30 from 28, it is seen that

$$\frac{4}{\tau}(1+\rho)\left(\frac{t}{t+1}\right)^{\eta} a_2 = x_1 - y_1,$$

$$a_2 = \frac{\tau(x_1 - y_1)}{4(1+\rho)\left(\frac{t}{t+1}\right)^{\eta}},$$

Applying triangle inequality and 32, one can get

$$|a_2| \le \frac{|\tau x_1|}{2(1+\rho)\left(\frac{t}{t+1}\right)^{\eta}},$$

Applying Lemma 1, one can get

$$|a_2| \le \frac{|\tau|}{2(1+\rho)\left(\frac{t}{t+1}\right)^{\eta}}. \tag{33}$$

Adding 29 and 31, it yields

$$\frac{6}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_2^2 = x_2 + y_2 + \frac{1}{2}(x_1^2 + y_1^2),$$

Applying triangle inequality and Lemma 1, it is seen that

$$\frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_2^2| \le 2 - |x_1|^2 - |y_1|^2 + \frac{1}{2}\left(|x_1^2| + |y_1^2|\right),$$

Again applying Lemma 1 leads to

$$\frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_2^2| \le 3,$$

Therefore

$$|a_2| \le \sqrt{\frac{|\tau|}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}}. \tag{34}$$

From 33 and 34, the first part of our theorem is fulfilled.

Subtracting 31 from 29, it is seen that

$$\frac{6}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_3 = \frac{6}{\tau}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} a_2^2 + x_2 - y_2 + \frac{1}{2}(x_1^2 - y_1^2),$$

Applying triangle inequality, lead to

$$\frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_3| \le \frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_2^2| + |x_2| + |y_2| + \frac{1}{2}\left(|x_1^2| + |y_1^2|\right),$$

Applying Lemma 1, it is seen that

$$\frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_3| \le \frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_2^2| + 2 - |x_1^2| - |y_1^2| + \frac{1}{2}\left(|x_1^2| + |y_1^2|\right),$$

$$\frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_3| \le \frac{6}{|\tau|}(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} |a_2^2| + 3,$$

$$|a_3| \le |a_2^2| + \frac{|\tau|}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}. \tag{35}$$

From 34 and 35, one can find

$$|a_3| \le \frac{|\tau|}{(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}}. \tag{36}$$

From 33 and 35, one can find

$$|a_3| \le \frac{|\tau|^2}{4(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}} + \frac{|\tau|}{2(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta}},$$

$$|a_3| \le \frac{|\tau|\left(|\tau|(1+2\rho)\left(\frac{t}{t+2}\right)^{\eta} + 2(1+\rho)^2\left(\frac{t}{t+1}\right)^{2\eta}\right)}{4(1+2\rho)(1+\rho)^2\left(\frac{t}{t+2}\right)^{\eta}\left(\frac{t}{t+1}\right)^{2\eta}}. \tag{37}$$

From 36 and 37, the second part of our theorem is fulfilled.

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Conclusion

In this research, two subclasses of bi-univalent functions were defined. The first subclass mentioned in Definition 3 was defined using the subordination of the shell-like curve functions related to Fibonacci numbers and the second subclass mentioned in Definition 4 was defined using the subordination of the exponential function. The Komatu integral operator was used in each of these subclasses. Limits were obtained for the elementary coefficients, specifically the second and third coefficients for these subclasses in Theorem 1 and Theorem 2.

Authors' Declaration

- Conflicts of Interest: None.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at the University of Tishreen.

Authors' Contribution Statement

This work was carried out in collaboration between all authors. M. A. conducted the study, produced the results, and wrote the manuscript. H. B. and M. A. collected and arranged the references. A.K.W., reviewed and proofread the manuscript, and communicated with the journal. All authors read and approved the final manuscript.

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فئات فرعية جديدة من الدوال ثنائية أحادية التكافؤ المرتبطة بالدوال الأسية وأرقام فيبوناتشي

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الخلاصة

تمت مناقشة الفئة Σ للدوال ثنائية أحادية التكافؤ من قبل الباحث لوين وحصل على تقدير للمعامل الثاني فيها، عرَّف ساكار و واناس فئتين فرعيتين جديدتين للدوال ثنائية أحادية التكافؤ وحصلا على الحدود العليا للمعاملات الأولية $|a_2|$ و $|a_3|$ للتوابع في هذه الفئات الفرعية، دزيوك وآخرون. قدموا الفئة SLM_α من الدوال الشبيهة بالصدفة المحدبة α ، والتي تشير إلى وجود اتصال بين الدالة $\tilde{p}(z)$ وأرقام فيبوناتشي. في الآونة الأخيرة تم تعريف العديد من فئات الدوال ثنائية أحادية التكافؤ، استنادًا إلى مؤثرات معروفة مثل مؤثر سيلاجين، ومؤثر تريمبلي، ومؤثر تكامل كوماتو، ومؤثر الالتواء، ومؤثر العبودي التفاضلي وغيرها. يهدف هذا البحث إلى تقديم فئتين فرعيتين جديدتين من الدوال ثنائية أحادية التكافؤ باستخدام التبعية ومؤثر تكامل كوماتو والتي تتضمن الدوال الأسية والمنحنيات الشبيهة بالصدفة مع أرقام فيبوناتشي، وكذلك إيجاد تقدير للمعاملات الأولية لهذه الفئات الفرعية. تم تعريف الفئة الفرعية الأولى باستخدام التبعية لتابع المنحنى الشبيهة بالصدفة المتعلق بأرقام فيبوناتشي وتم تعريف الفئة الفرعية الثانية باستخدام التبعية للدالة الأسية. وتم استخدام مؤثر تكامل كوماتو في كل فئة من هذه الفئات الفرعية. تم الحصول على الحدود للمعاملات الأولية، وتحديداً المعاملين الثاني و الثالث لتوابع هذه الفئات الفرعية.

الكلمات المفتاحية: الدوال ثنائية أحادية التكافؤ، حدود المعاملات، الدالة الأسية، أرقام فيبوناتشي، مؤثر تكامل كوماتو، التبعية.