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New Formulas of Special Singular Matrices

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Abstract:

Many of the elementary transformations of determinants which are used in their evaluation and in the solution of linear equations may by expressed in the notation of matrices. In this paper, some new interesting formulas of special matrices are introduced and proved that the determinants of these special matrices have the values zero. All formulation has been coded in **MATLAB 7.**

Keywords: singular matrices, determinant

Introduction

From the beginning of the history of matrix theory , matrices and determinants have been closely connected . Indeed , when Sylvester first used the word "matrix", it was to define an "oblong arrangement of terms "out of which determinants could be formed by "selecting at will n lines and m columns" [1] .

The determinant was rediscovered , and much was written on the subject between 1750 and 1900. during this era. Determinants became the major tool used to analyze and solve linear systems. The study and use of determinants eventually gave way to cayley's matrix algebra, and today matrix and linear algebra are in the main stream of applied mathematics [2],[3].

Many authors and researchers studied the matrices and determinants . Chien and Sinclair [4] gave an approach to characterization of non commutative algebras by means of the polynomial identities and computation the determinant . Johnson and Olesky [5]. Presented new factorization results that generate all matrices with positive determinant . while Vandenberghe and Boyd .[6] , described the problem of maximizing the determinant of a matrix subject to linear matrix inequalities. This paper will begin with the definition of matrices and determinant

Definitions

It should be understood that in terms of this paper the determinant is only defined for square matrices. If a matrix is not square then its determinant does not exist. With that in mind we begin with the geometric definition of determinant and progress to the classical algebraic definition of determinant. These definitions apply whether the matrix has numerical or symbolic entries.

1. The algebraic definition of determinant [7]

A set of m x n numbers, real or complex, arranged in an array of m columns and n rows is called a matrix, thus

$$\begin{matrix} a_{11} & a_{12} & a_{13}\dots & a_{1m} \\ a_{21} & a_{22} & a_{23}\dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3}\dots & a_{nm} \end{matrix}$$

Is a matrix when m = n we speak of a square matrix of order n. the

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matrix written above, with m = n is associated with determinant

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix}$$

There are many different ways of defining a determinant of order n, though all the definitions lead to the same result in the end.

We now define a determinant of order n [8], the determinant

$$\Delta_n = \begin{vmatrix} a_1 & b_1 & \cdots & j_1 & k_1 \\ a_2 & b_2 & : & j_2 & k_2 \\ \vdots & \vdots & : & \vdots & \vdots \\ a_n & b_n & : & j_n & k_n \end{vmatrix} \quad (1)$$

Is that function of the a's , b's , ……. , k's which satisfies the three conditions :

(i) It is an expression of the form

$$\sum \mp a_r\, b_s \ldots . k_\theta, \ldots (2)$$

(ii) The leading diagonal tern, $a_1\, b_2$ ….. k_n , is prefixed by the sign + .

(iii) The sign prefixed to any other term in such that the interchange of any two letters throughout (2) reproduces the same set of terms, but in different order of occurrence, and with the opposite signs prefixed.

2. The geometric definition of determinant[3]

The most intuitive definition of determinant it is the geometric definition . It is this definition that is often overlooked and rarely used for computation . We mention it here for completeness and in the hope that a visual picture may aid in the understanding and usage of the determinant .

We will begin with a simple l x l matrix . In this case the determinant of the matrix is the signed length of the line from the origin to the point on the number line marked by the entry of the matrix .

So if the single entry of the matrix is positive , we consider the determinant to be the length of the line from the origin to the point going in the positive `x` direction . If the entry is negative then the determinant is the negative of the length of the line from the origin to the point going in the negative `x` direction .

In the case of a 2 x 2 matrix we look at the matrix as set of two points in the Euclidean plane . Using these two points we make a parallelogram that includes the origin . The determinant is then the signed area of the parallelogram. For example if the matrix was:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we would have a rectangle with corner points at (0,0), (2,0), (2,1) and (0,1) . And the determinant would be (positive) 2.

For a 3x3 matrix the concept is much the same. We consider the matrix to be 3 points in 3-dimentional Euclidean space.We create a parallelepiped that includes the three points and the origin. The determinant is then the signed volume of the parallelepiped .

This concept extends to the higher dimensions of Euclidean space . So the determinant of an n x n matrix would be the ·volume· of the *n*-dimensional parallelepiped formed from the n points of the matrix .

3.Elementary properties and operations of determinants.

Some of the theorems concerning the properties of determinants are given in this section [9],[10],[11].

Theorem(1):
If a determinants has two columns, or two rows, identical , its value is zero.

Theorem (2):
The value of a determinant is unaltered if to each element of on column (or row) is added a constant multiple of the corresponding element of another column (or row) ; in particular

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 + \lambda b_1 & b_1 & c_1 \\ a_2 + \lambda b_2 & b_2 & c_2 \\ a_3 + \lambda b_3 & b_3 & c_3 \end{vmatrix}$$

Remark: there are many extensions of theorem (2) for example , by repeated applications of theorem (2) it can be proved theorems (3) and (4)

Theorem (3):
The value of a determinant is unaltered if we add to each column (or row) fixed multiples the subsequent columns (or rows); in particular

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{vmatrix} a_1 + \lambda b_1 + \mu c_1 & b_1 + v c_1 & c_1 \\ a_2 + \lambda b_2 + \mu c_2 & b_2 + v c_2 & c_2 \\ a_3 + \lambda b_3 + \mu c_3 & b_3 + v c_3 & c_3 \end{vmatrix}$$

Theorem (4):
The value of a determinant is unaltered if we use add multiples of any one column (or row) to every other column (or row) ; in particular

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda b_1 & b_1 & c_1 + \mu b_1 \\ a_2 + \lambda b_2 & b_2 & c_2 + \mu b_2 \\ a_3 + \lambda b_3 & b_3 & c_3 + \mu b_3 \end{vmatrix}$$

4.New special singular matrices.

The theories of the matrix and of its determinant are closely knit together . Some new interesting formula of singular matrices are proposed and proved in this section.

Theorem (5) :
The value of a determinant for the following matrix is equal to zero.

$$\Delta = \begin{vmatrix} a & a+\lambda & .. & a+b\lambda \\ a+\lambda & a+2\lambda & .. & a+(b+1)\lambda \\ a+2\lambda & a+3\lambda & .. & a+(b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ a+b\lambda & a+(b+1)\lambda & .. & a+2b\lambda \end{vmatrix}$$

For n=3,4,5,………etc
Proof:
The determinant Δ can be rewritten in the following form:

$$\begin{vmatrix} a & a & .. & a \\ a & a & .. & a \\ a & a & .. & a \\ \vdots & \vdots & \vdots & \vdots \\ a & a & .. & a \end{vmatrix} + \begin{vmatrix} 0 & \lambda & .. & b\lambda \\ \lambda & 2\lambda & .. & (b+1)\lambda \\ 2\lambda & 3\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ b\lambda & (b+1)\lambda & .. & 2b\lambda \end{vmatrix}$$

Since every a_{ij} in the first matrix be changed to $a.a_{ij}$, every term in the determinant A is multiplied by a^n . Hence $|aA| = a^n|A|$. Where

$$|A| = \begin{vmatrix} 1 & 1 & .. & 1 \\ 1 & 1 & .. & 1 \\ 1 & 1 & .. & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & .. & 1 \end{vmatrix}$$

and hence $|A| = 0$

The determinant of the second matrix is envisaged by theorem (2), therefore

$$\begin{vmatrix} 0 & \lambda & .. & b\lambda \\ \lambda & 2\lambda & .. & (b+1)\lambda \\ 2\lambda & 3\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ b\lambda & (b+1)\lambda & .. & 2b\lambda \end{vmatrix} = \begin{vmatrix} 0-\lambda & \lambda & .. & b\lambda \\ \lambda-2\lambda & 2\lambda & .. & (b+1)\lambda \\ 2\lambda-3\lambda & 3\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ b\lambda-(b+1)\lambda & (b+1)\lambda & .. & 2b\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & \lambda & .. & b\lambda \\ -\lambda & 2\lambda & .. & (b+1)\lambda \\ -\lambda & 3\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda & (b+1)\lambda & .. & 2b\lambda \end{vmatrix}$$

Again with the aid of the result in theorem (2) ,we obtain:

$$\begin{vmatrix} 0 & \lambda & .. & b\lambda \\ \lambda & 2\lambda & .. & (b+1)\lambda \\ 2\lambda & 3\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ b\lambda & (b+1)\lambda & .. & 2b\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -\lambda & .. & b\lambda \\ -\lambda & -\lambda & .. & (b+1)\lambda \\ -\lambda & -\lambda & .. & (b+2)\lambda \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda & -\lambda & .. & 2b\lambda \end{vmatrix}$$

In virtue of theorem (1), the determinant of the above matrix has the value zero . Hence Δ has the value zero.

An extension of theorem (5) is given in theorem (6).

Theorem(6):

The value of a determinant of the following matrix is equal to zero:

$$\Delta = \begin{vmatrix} a & a+\lambda & .. & a+b\lambda \\ a+\mu & a+\mu+\lambda & .. & a+\mu+b\lambda \\ a+2\mu & a+2\mu+\lambda & .. & a+2\mu+b\lambda \\ \vdots & \vdots & \vdots & \vdots \\ a+b\mu & a+b\mu+\lambda & .. & a+b\mu+b\lambda \end{vmatrix}$$

For n=3,4,5……etc .

Proof : the same way as in theorem (5)

Lemma(1): The difference between the entries of the main diagonal and the entries of the secondary diagonal is equal to zero .

Examples:

For Theorem (5) let a= - 4, λ=7 , so the determinant has been expressed as :

$$|A| = \begin{vmatrix} -4 & 3 & 10 & 17 \\ 3 & 10 & 17 & 24 \\ 10 & 17 & 24 & 31 \\ 17 & 24 & 31 & 38 \end{vmatrix} = 0$$

For Theorem (6) let a= 3, μ=-9, λ=6 , so the determinant has been expressed as :

$$|A| = \begin{vmatrix} 3 & 9 & 15 & 21 \\ -6 & 0 & 6 & 12 \\ -15 & -9 & -3 & 3 \\ -24 & -18 & -12 & -6 \end{vmatrix} = 0$$

Discussion

Determinants are the major tool used to analyze and solve linear system . They are defined only for square matrices , and are scalars . They are a very important role in determining whether a matrix is invertible, and what the inverse is .

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صيغ جديدة لمصفوفات منفردة خاصة

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الخلاصة:

الكثير من التحولات الاساسية للمحددات التي تستخـــدم في حل المعادلات الخطيـــة يمكـــن ان يعبر عنهـا كملاحظات خاصة بهذه المصفوفات . في هذه الدراسة قدمت بعض الصيغ المهمة الخاصة بالمصفوفات واثبتت ان المحددات الخاصة بها تســاوي صفر. وقد تم استخدام برنامج الحاسبة الخاص بالمصفوفات للحل .