The Classical Continuous Optimal Control for Quaternary Nonlinear Parabolic System

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Abstract:
In this paper, our purpose has been to study the classical continuous optimal control problem for the quaternary nonlinear parabolic boundary value problem. Under suitable assumptions and with the given quaternary classical continuous control vector, the existence and uniqueness theorem for the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem has been stated and proved via the Galerkin Method, and the first compactness theorem. Furthermore, the continuity operator between the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem and the corresponding quaternary classical continuous control vector have been proved. The existence of a quaternary classical continuous optimal control vector has been stated and proved under suitable conditions.

Keywords: Classical Optimal Control, Cost Function, Galerkin Method, Lipschitz continuity, Parabolic Boundary Value Problems.

Introduction:
Optimal control problem (OCP) means endogenously controlling a parameter in a mathematical model to produce an optimal (cost) output. The problem comprises an objective (or cost) function, which is a function of the state and control variables, and the constraints on the control. The problem seeks to optimize the objective function subject to the constraints construed by the model describing the evolution of the underlying system. Optimal control problems (OCPs) play an important role in many practical applications, such as in weather conditions, economics, robotics, aircraft, medicine, and many other scientific fields. They are two types of OCPs; the classical and the relax type, each one of these two types is dominated either by nonlinear ODEs or by nonlinear PDEs (NLPDEs). The classical continuous optimal control problem (CCOCP) dominated by nonlinear parabolic or elliptic or hyperbolic PDEs are studied in respectively (resp.). Later, the study of the CCOCPs dominated by the three types of nonlinear PDEs is generalized in to deal with CCOCPs dominated by coupling NLPDEs of these types resp., and then these studies are generalized also to deal with CCOCPs dominated by triple and NLPDEs of the three types.

In each type of these CCOCPs the problem consists of; an initial or a boundary value problem (the dominating equations), the objective (cost) function of the classical continuous control vector, the state vector, and the constraints on the control vector. The study in each one of these problems included; the state and proves the existence of a unique “ state “ vector solution for the weak form (WF) which is usually obtained from the initial or the boundary value problem (the dominating equations) where the classical continuous control is known. The continuity of the Lipschitz operator between the state vector solution of the WF for the dominating equations and the corresponding classical continuous control vector is proved. The existence theorem of a classical continuous optimal control vector for the problem is stated and proved under suitable conditions. All of the above-mentioned studies encouraged us to think about generalizing the study of the CCOCP dominated by triple NLPDEs of parabolic type to a CCOCP.
dominated by quaternary nonlinear parabolic boundary value problem (QNLPBVP). According to this idea for the generalization, the mathematical model for the dominating equation is needed to be found, as well as the cost function, the spaces of definition for the control, and the state vectors, which all of them are needed to be generalized. Hence all the theorems and the results which were accompanied in the above studies for the CCOCPs dominated by the couple and triple nonlinear PDEs of the three types are needed to be stated and proved, for the “new” proposed CCOCP which is dominating by the QNLPPDEs.

The study of the proposed CCOCP is very interesting in the field of applied mathematics because the proposed model represents a generalization of the heat equation from one side, and the other, it represents a multi objectives problem that has many applications. Furthermore, the results of this paper are very useful, because give the green light about the ability for solving the problem numerically.

The study of the CCOCP dominated by the QNLPBVP which is proposed in this paper starts with the state and proof of the existence theorem of the quaternary state vector solution (QSVS) of the weak form (WF) for the QNLPBVP using the Galerkin Method (GM) with the first compactness theorem, under suitable conditions and when the quaternary classical continuous control vector (QCCCV) is known. The continuity of the Lipschitz operator between the state vector solution of the WF for the QNLPBVP and the corresponding existence theorem of the Lipschitz continuous control vector are proved. The existence theorem of a quaternary classical continuous optimal control vector (QCCOCV) is stated and demonstrated under suitable conditions.

**Problem Description:**

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and regular region with Lipschitz boundary $\Gamma = \partial \Omega$, $x = (x_1, x_2)$, $Q = I \times \Omega$, $I = [0, T]$, $\Gamma = \partial \Omega$, $\Sigma = \Gamma \times 1$. The CCOCP consists of the following QNLPBVP, which are constructed by us, are given by (with $y_i = y_i(x, t)$ and $u_i = u_i(x, t)$, $\forall i = 1, 2, 3, 4$):

1. $y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x, t, y_1, u_1)$, in $Q$
2. $y_{2t} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x, t, y_2, u_2)$, in $Q$
3. $y_{3t} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x, t, y_3, u_3)$, in $Q$
4. $y_{4t} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x, t, y_4, u_4)$, in $Q$

With the following boundary conditions (BCs) and initial conditions (ICs):

$y_i(x, t) = 0$, $\forall i = 1, 2, 3, 4$.

on $\Sigma$

$y_i(x, 0) = y^0_i(x)$, $\forall i = 1, 2, 3, 4$.

on $\Omega$

Where $\tilde{y} = (y_1, y_2, y_3, y_4) \in (H^2(\Omega))^4$ is the QSVS, $\tilde{u} = (u_1, u_2, u_3, u_4) \in (L^2(\Omega))^4$ is the QCCCV and $(f_1, f_2, f_3, f_4) \in (L^2(\Omega))^4$ is given, for all $x \in \Omega$.

The Set of Admissible Control is:

$\mathcal{W}_A = \{\tilde{w} \in (L^2(\Omega))^4 | \tilde{w} \in \tilde{U} \subset \mathbb{R}^4 \text{ a.e. in } Q\}$.

The Cost Function is:

$G_0(\tilde{u}) = \int_Q g_{01}(x, t, y_1, u_1) dx dt + \int_Q g_{02}(x, t, y_2, u_2) dx dt + \int_Q g_{03}(x, t, y_3, u_3) dx dt + \int_Q g_{04}(x, t, y_4, u_4) dx dt$

Where $\tilde{y} = \tilde{y}_I$ is the QSVE of Eq.1–Eq.6 corresponding to the QCCCV $\tilde{u}$.

The classical continuous optimal control problem is to find $\tilde{u} \in \mathcal{W}_A$, s.t.:

$G_0(\tilde{u}) = \min_{\tilde{w} \in \mathcal{W}_A} G_0(\tilde{w})$.

The notations $(v, v)$ and $\|v\|_{L^2(\Omega)}$ are referred to as the inner product and the norm in $L^2(\Omega)$, the notations $(\tilde{v}, \tilde{v})_{\mathcal{H}^1(\Omega)}$ and $\|\tilde{v}\|_{\mathcal{H}^1(\Omega)}$ the inner product and the norm in $\mathcal{H}^1(\Omega)$, of $\tilde{v}$ and $\tilde{v}$.

$\tilde{v}^*$ is the dual of $\tilde{v}$, $L^2(I, \mathcal{V}) = (L^2(I, \mathcal{V}))^*$, $L^2(I, \mathcal{V}^*) = (L^2(I, \mathcal{V}^*))^*$, $L^2(Q) = (L^2(\Omega))^4$, and $(L^2(\Omega))^*$ it is dual.

The Weak Form of the Quaternary State Vector Equations:

The WF of Eq.1–Eq.6 (with $\tilde{y} \in \mathcal{H}^1_0(\Omega)$) is:

$\langle y_{1t} \rangle + \langle y_{2t} \rangle + \langle y_{3t} \rangle + \langle y_{4t} \rangle = \langle f_1 \rangle + \langle f_2 \rangle + \langle f_3 \rangle + \langle f_4 \rangle$

$\langle y_{1t} \rangle + \langle y_{2t} \rangle + \langle y_{3t} \rangle + \langle y_{4t} \rangle = \langle f_1 \rangle + \langle f_2 \rangle + \langle f_3 \rangle + \langle f_4 \rangle$
\(\langle y_{dt}, v_t \rangle + \langle V y_t, \nabla v_t \rangle - \langle y_t, v_t \rangle + \langle y_{22}, v_{22} \rangle - \langle y_{23}, v_{23} \rangle + \langle y_{24}, v_{24} \rangle = f_t(y_t, u_t, v_t), \quad 14 \)

The following assumptions, propositions, and Lemmas are important in our study of the QCCOC.

**Assumptions A:** Suppose for \((x, t) \in Q\), and \(y_t, y_{ti}, u_t \in \mathbb{R}, \forall i = 1, 2, 3, 4\) that:

(i) \(f_t\) is Carathéodory type (Cara. T.) on \(Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) and the following conditions w.r.t. \(y_t \& u_t\) are held: \(|f_t(x, t, y_t, u_t)| \leq \eta_t(x, t) + c_1|y_t| + c_\zeta|u_t|\), where \(\eta_t \in L^2(Q, \mathbb{R})\), and \(c_\zeta, c_1 > 0\).

(ii) \(fi\) satisfies Lipschitz condition w.r.t. \(y_t\), i.e.: 
\[|f_t(x, t, y_t, u_t) - f_t(x, t, y_{ti}, u_t)| \leq L'(|y_t - y_{ti}|), \quad L' > 0.\]

**Proposition 1 16:**

Let \(f : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^m\) is Cara. T., let \(F\) be a functional, s.t. \(F(y) = \int_\Omega f(x, y(x)) \, dx\), where \(\Omega\) is a measurable subset of \(\mathbb{R}^n\), and suppose that 
\[|f(x, y)| \leq \zeta(x) + \eta(x)|y|^{l_2}\], \(\forall(x, y) \in \Omega \times \mathbb{R}^n\), \(y \in L^p(\Omega \times \mathbb{R}^n)\). Where \(\zeta \in L^1(\Omega \times \mathbb{R}), \exists L^p(\Omega \times \mathbb{R}), \quad \alpha \in [0, p]\), if \(p \in [1, \infty]\), and \(\eta \equiv 0\), if \(p = \infty\). Then, \(F\) is continuous on \(L^p(\Omega \times \mathbb{R})\).

**Lemma 1 19:**

Let \(V, H, V^*\) be three Hilbert spaces, each space included in the following one in the following equality,

\[\|u\|_{L^2(\Omega)} = (\int_\Omega u(x)^2 \, dx)^{1/2}\] or \(\|u\|_{L^\infty(\Omega)} = \text{ess.sup.} |u(x)|\).

\(V^*\) being the dual of \(V\). If a function \(u\) belongs to \(L^2(0, T; V)\) and it’s derivative \(u\) belongs to \(L^2(0, T; V^*)\), then \(u\) is almost everywhere (a.e.) equal to a function continuous from \([0, T]\) in to \(H\) and one have the following equality, which holds in the scalar distribution sense on \((0, T)\):

\[\frac{d}{dt} |u(t)|^2 \leq 2\langle \dot{u}(t), u(t) \rangle, \quad u \in U, U \subset \mathbb{R} \text{ is compact.}

**Assumption B:**

Consider \(g_{01}\) (for each \(i = 1, 2, 3, 4\)) is of Cara. T. on \(Q \times \mathbb{R}^4\) and the following conditions w.r.t. \(y_t, u_t\), hold:

\[g_{01}(x, t, y_t, u_t) \leq \eta_{01}(x, t) + c_{01}(y_t)^2 + c_{02}(u_t)^2, \quad \forall y_t, u_t \in \mathbb{R} \text{ and } \eta_{01} \in L^2(Q).\]

**Lemma 2 13:**

Let \(g : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}\) is of Cara. T. on \(\Omega \times \mathbb{R}^4\), and satisfies:

\[|g(x, y, u)| \leq \eta(x) + cy^2 + cu^2, \quad \text{where } \eta \in L^1(\Omega, \mathbb{R}), c \geq 0 \quad \text{and} \quad c \geq 0.\]

Then, \(\int_\Omega g(x, y, u) \, dx\) is continuous on \(L^2(\Omega, \mathbb{R}^2)\), with \(u \in U, U \subset \mathbb{R}\) is compact.

**Main Results:**

**The Solution of the Weak Form:**

This section deals with the unique and existence uniqueness theorem of QSVS for the weak form of the QNLBPV under the suitable assumptions when the QCCCV is given.

**Theorem 1 (The Existence and Uniqueness Theorem for the Weak Form):**

With Assumptions A, for each given QCCCV \(v_t \in L^2(Q)\), the weak form of Eq.8 – Eq.15 has a unique QSVS \(y_t^* \in L^2(I, \mathbb{V})\), with \(y_t \in L^2(I, \mathbb{V})\).

**Proof:**

Let for any \(n, \widetilde{V}_n = V_n \times V_n \times V_n \times V_n\) \(\forall n \in \mathbb{V}\) (i.e. each subspace \(V_n\) has its different basis) be the set of piecewise affine functions in \(\Omega\), let \(\widetilde{V}_n = (V_n, V_n, V_n, V_n)\) with \(v_n \in V_n (\forall i = 1, 2, 3, 4)\) and \(\widetilde{V}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})\). Let \(\mathcal{Y}_n\) be an approximation for \(y_t\), s.t.:

\[y_n = \sum_{i=1}^n c_{ij}(t)v_i(x), \quad \forall i = 1, 2, 3, 4.\]

Where \(c_{ij}(t)\) is an unknown function of \(t\), \(\forall i \in \{1, 2, 3, 4\}\).

Hence, Eq.8 – Eq.15 become \(\forall v_t \in V_n, i = 1, 2, 3, 4\):

\[\langle y_{1nt}, v_t \rangle + \langle y_{1nt}, v_{1t} \rangle + \langle y_{1n}, v_{1t} \rangle - \langle y_{1n}, v_n \rangle + \langle y_{2nt}, v_{2t} \rangle + \langle y_{2nt}, v_{2t} \rangle - \langle y_{2n}, v_{2t} \rangle + \langle y_{3nt}, v_{3t} \rangle + \langle y_{3nt}, v_{3t} \rangle - \langle y_{3n}, v_{3t} \rangle + \langle y_{4nt}, v_{4t} \rangle + \langle y_{4nt}, v_{4t} \rangle - \langle y_{4n}, v_{4t} \rangle = f_n.\]

By utilizing Eq.16 (for \(i=1, 2, 3, 4\)) in Eq.17 – Eq.24 resp., and then setting \(v_t = v_{it}, \forall i = 1, 2, 3, 4\) and \(i = 1, 2, 3, 4, \) one gets that Eq.17 – Eq.24 are equivalent to the following nonlinear system of 1 st order ODEs with ICs, which has a unique solution (from the continuity of all the matrices and vectors):

\[A_1 C_1(t) + B_1 C_1(t) - D_2 C_1(t) + EC_3(t) + FC_4(t) = b_1 \left( \widetilde{V}_1^T(x) C_1(t) \right), \quad 25\]

\[A_1 C_1(0) = b_1, \quad 26\]

\[A_2 C_2(t) + B_2 C_2(t) + KC_1(t) - MC_3(t) - NC_4(t) = b_2 \left( \widetilde{V}_2^T(x) C_2(t) \right), \quad 27\]

\[A_2 C_2(0) = b_2, \quad 28\]

\[A_3 C_3(t) + B_3 C_3(t) - PC_1(t) + QC_2(t) + SC_4(t) = b_3 \left( \widetilde{V}_3^T(x) C_3(t) \right), \quad 29\]

\[A_3 C_3(0) = b_3, \quad 30\]

\[A_4 C_4(t) + B_4 C_4(t) - XC_1(t) + YC_2(t) - ZC_3(t) = b_4 \left( \widetilde{V}_4^T(x) C_4(t) \right), \quad 31\]
\[ A_4C_4(0) = b_0. \]

Where \( l = 1, 2, \ldots, n, \forall i = 1, 2, 3, 4, \]
\[ A_i = (a_{ij})_{n \times n}, \quad a_{ij} = (v_{ij}, v_{il}), \quad B_i = (b_{ij})_{n \times n}, \]
\[ b_{ij} = \langle v_{ij}, v_{il} \rangle, \quad D_i = (d_{ij})_{n \times n}, \]
\[ d_{ij} = (v_{ij}, v_{jl}). \]

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Since \( y_0^0 = \hat{y}_0^0(x) \in L^2(\Omega) \), then there is a sequence \( \{v_{in}\} \) with \( v_{in} \in V_n, \) s.t.
\[ \lim_{n \to \infty} ||y_0^0 - v_{in}||_{L^2(\Omega)} = 0. \]

\[ \lim_{n \to \infty} \|y_0^0 - v_{in}\|_{L^2(\Omega)} = 0, \quad \forall v_{in} \in V_n. \]

Similarly, once get that \( \lim_{n \to \infty} ||y_0^0||_{L^2(\Omega)} = b_1, \)
\[ \lim_{n \to \infty} \|y_0^0\|_{L^2(\Omega)} = b_1, \quad \forall v_{in} \in V_n. \]

Thus, \( y_0^0 \rightarrow y_1^0 \) ST in \( L^2(\Omega) \) with \( \lim_{n \to \infty} ||y_0^0||_{L^2(\Omega)} = b_1. \)

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The Convergence of the Solution:

Let \( \{\bar{V}_n\}_{n=1}^\infty \) be a sequence of subspaces of \( \bar{\mathcal{V}}, \) s.t.
\[ \forall \bar{v} \in \bar{\mathcal{V}}, \] there is a sequence \( \{\bar{v}_n\} \) with \( \bar{v}_n \in \bar{V}_n, \forall n, \) and \( \bar{v}_n \to \bar{v} \) converges strongly in \( \bar{\mathcal{V}} \) (which gives \( \bar{v}_n \to \bar{v} \) converges strongly in \( L^2(\Omega) \),

For each \( n, \) with \( \bar{V}_n \subset \bar{\mathcal{V}}, \) problem Eq.17 – Eq.24 has a unique solution \( \hat{y}_n, \) hence corresponding to the sequence of subspaces \( \{\bar{V}_n\}_{n=1}^\infty \), there is a sequence of approximation problems like Eq.17 – Eq.24, now by setting \( \bar{v} = \bar{v}_n, \) for \( n = 1, 2, \ldots, \) one gets that
\[ \forall \bar{v}_n \in \bar{V}_n: \]
\[ (y_{1ntv}, y_{1tv}) + (y_{1ntv}, y_{1tv}) + (y_{2ntv}, y_{2tv}) - (f_2(y_{2ntv}, u_{2tv}), y_{2tv}) + (f_3(y_{3ntv}, u_{3tv}), y_{3tv}) + (f_4(y_{4ntv}, u_{4tv}), y_{4tv}) dt = \]

Since \( \hat{y}_{nt} \in L^2(I, V^*) \) and \( \hat{y}_n \in L^2(I, V) \) then, Lemma 1 can be used for the \( 1^{st} \) term in L.H.S. of Eq.33, on the other hand, since the \( 2^{nd} \) term is non-negative taking \( T = t, \) with \( t \in I, \) finally using Assumptions A-i for the R.H.S. of Eq.33, it yields to:

\[ \int_0^t \frac{d}{dt} ||\hat{y}_{nt}(t)||^2_{L^2(\Omega)} dt \leq ||\hat{y}_1(t)||^2_{L^2(\Omega)} + ||\hat{y}_2(t)||^2_{L^2(\Omega)} + \]
\[ ||\hat{y}_3(t)||^2_{L^2(\Omega)} + ||\hat{y}_4(t)||^2_{L^2(\Omega)} + \epsilon_1||u_1(t)||^2_{L^2(\Omega)} + \]
\[ \epsilon_2||u_2(t)||^2_{L^2(\Omega)} + \]
\[ \epsilon_3||u_3(t)||^2_{L^2(\Omega)} + \epsilon_4||u_4(t)||^2_{L^2(\Omega)} + c_0 \int_0^t ||\hat{y}_n||^2_{L^2(\Omega)} dt, \]
\[ c_0 = \max(c_5, c_6, c_7, c_8). \]
\[
(y_{3n}^0, v_{3n}^0) = (y_{3n}^0, v_{3n}^0), \\
(y_{3n}^0, v_{3n}^0), (y_{3n}^0, v_{3n}^0) - (y_{3n}^0, v_{3n}^0) + (y_{3n}^0, v_{3n}^0) = (y_{3n}^0, v_{3n}^0).
\]

Then Eq. 34 – Eq. 41 has a sequence of solutions \{\tilde{y}_n\}_{n \in \mathbb{N}} , but from the previous steps once has \|\tilde{y}_n\|_{L^2(\Omega)}^2 and \|\tilde{v}_n\|_{L^2(\Omega)}^2 are bounded, then by Alaoglu’s theorem, there is a subsequence of \{\tilde{y}_n\}_{n \in \mathbb{N}} , for simplicity say again \{\tilde{y}_n\}_{n \in \mathbb{N}} , s.t. \tilde{y}_n \rightarrow \tilde{y} converges weakly in \(L^2(Q)\) and \tilde{y}_n \rightarrow \tilde{y} converges weakly in \(L^2(I, \nu)\).

At this point, it’s necessary to demonstrate that the norm \|\tilde{y}_n\|_{L^2(I, \nu)}^2 is bounded, the demonstration of this point will be left here and it will be shown later in Theorem 3, so suppose it is bounded, and since

\[
(L^2(\mathbb{R}, V))^4 \subset (L^2(\mathbb{R}, \Omega))^4 \subset (L^2(\mathbb{R}, V))^4 \subset (L^2(\mathbb{R}, V))^4
\]

As a result, the injections of \((L^2(\mathbb{R}, V))^4\) into \((L^2(\mathbb{R}, \Omega))^4\), and of \((L^2(\mathbb{R}, \Omega))^4\) into \((L^2(\mathbb{R}, V))^4\) are continuous, the injection of \((L^2(\mathbb{R}, V))^4\) into \((L^2(Q))^4\) is compact. From Assumptions A, the Cauchy-Schwarz inequality, then the first compactness theorem can be applied here to obtain that there is a subsequence of \{\tilde{y}_k\} say again \{\tilde{y}_k\} such that \tilde{y}_k \rightarrow \tilde{y} converges strongly in \(L^2(Q)\).

Now, consider the weak form Eq. 34 – Eq. 41 and take any arbitrary \(v_i \in V, \) so there is a sequence \{v_{in}\}, \(v_{in} \in V, \) \(\forall n \) s.t. \(v_{in} \rightarrow v_i\ converges strongly in V\), then \(v_{in} \rightarrow v_i\ converges strongly in V\)), \(\forall i = 1, 2, 3, 4\).

Now, multiplying both sides of Eq. 34, Eq. 36, Eq. 38, and Eq. 40 by \(\phi_i \in C^1(0, T), \forall i = 1, 2, 3, 4\) respectively, with \(\phi_i(T) = 0, \phi_i(0) \neq 0, \) then integrating both sides w.r.t. \(t\) from \(0\) to \(T\), and then integrating by parts the 1st terms in L.H.S. of each equality, one has:

\[
\int_0^T (y_{3n}^0, v_{3n}^0) \phi_1(t) \ dt + \int_0^T (\mathcal{Y}_{3n}^0, v_{3n}^0) \phi_1(t) \ dt + \int_0^T (y_{3n}^0, v_{3n}^0) \phi_1(t) \ dt + \int_0^T (y_{3n}^0, v_{3n}^0) \phi_1(t) \ dt = \int_0^T (f_0(y_{3n}^0, v_{3n}^0) \phi_1(t) \ dt + (y_{3n}^0, v_{3n}^0) \phi_1(0).43
\]

\[
\int_0^T (y_{2n}^0, v_{2n}^0) \phi_2(t) \ dt + \int_0^T (y_{2n}^0, v_{2n}^0) \phi_2(t) \ dt - \int_0^T (y_{2n}^0, v_{2n}^0) \phi_2(t) \ dt = \int_0^T (f_0(y_{2n}^0, u_{2n}^0, v_{2n}^0) \phi_2(t) \ dt + (y_{2n}^0, v_{2n}^0) \phi_2(0).44
\]

\[
\int_0^T (y_{3n}^0, v_{3n}^0) \phi_3(t) \ dt - \int_0^T (y_{3n}^0, v_{3n}^0) \phi_3(t) \ dt = \int_0^T (f_0(y_{3n}^0, v_{3n}^0) \phi_3(t) \ dt + (y_{3n}^0, v_{3n}^0) \phi_3(0).45
\]

\[
\int_0^T (f_0(y_{4n}^0, v_{4n}^0) \phi_4(t) \ dt + \int_0^T (f_0(y_{4n}^0, v_{4n}^0) \phi_4(t) \ dt = \int_0^T (f_0(y_{4n}^0, v_{4n}^0) \phi_4(t) \ dt + (y_{4n}^0, v_{4n}^0) \phi_4(0).46
\]
\( (y^0_{in}, v_{in})\varphi_4(0) \rightarrow (y^0_{v}, v_4)\varphi_4(0) \).

On the other hand, since \( v_{in} \in V_n \), then \( w_{in} = v_{in}\varphi_i \in C[Q] \) for \( i = 1, 2, 3, 4 \), and \( w_{in} \rightarrow w_i = v_i\varphi_i \) converges strongly in \( L^2(Q) \), thus \( w_{in} \) is measurable w.r.t. \( (x, t) \), and then using Assumptions A-i, and Proposition 1, once has \( \int_0^T(f_i(y_{in}, u_{in}) w_{in}) \, dx \, dt \) is continuous w.r.t. \( (y_{in}, u_{in}, w_{in}) \), since \( y_{in} \rightarrow \tilde{y} \) converges strongly in \( L^2(Q) \), therefore

\[ f_0^T(f_i(y_{in}, u_{in}) \varphi_i(t) \, dt \rightarrow \int_0^T(f_i(y_{in}, u_{in}) \varphi_i(t) \, dt, \] for all \( \varphi_i \in C[Q] \).

From the above converges Eq.43 – Eq.54, give:

\[ -\int_0^T(y^0_{1}, v_1)\varphi_1(t) \, dt + \int_0^T([\nabla y_1, \nabla v_1] + (y^0_{1}, v_1))\varphi_1(t) \, dt + \int_0^T(y^0_{2}, v_2)\varphi_2(t) \, dt = f_0^T(f_2(y^0_{1}, u_{in}) v_2(t) + (y^0_{2}, v_2) \varphi_2(t) + f_0^T(f_3(y^0_{1}, u_{in}) v_3(t) + (y^0_{3}, v_3) \varphi_3(t) + f_0^T(f_4(y^0_{1}, u_{in}) v_4(t) + (y^0_{4}, v_4) \varphi_4(t) \, dt, \] for all \( \varphi_i \in C[Q] \).

Adding them together and then, integrating both sides of them from zero to \( T \), then, adding the four obtained equalities together and then, integrating both sides of the obtained equations from zero to \( T \), to obtain that:

\[ f_0^T(y^{\ast}_{in}, \tilde{y}_n) \, dt + f_0^T(y_n(t) \varphi_2(t) \, dt = f_0^T(f_1(y^0_{1}, u_{in}) \varphi_2(t) + f_0^T(f_2(y^0_{1}, u_{in}) \varphi_3(t) + f_0^T(f_3(y^0_{1}, u_{in}) \varphi_4(t)), \] for all \( \varphi_i \in C[Q] \).

On the other hand substituting \( v_i = y_{in} \), \( \forall i = 1, 2, 3, 4 \), in Eq.17, Eq.19, Eq.21 and Eq.23 resp. then, adding the four obtained equalities together and then, integrating both sides of the obtained equations from zero to \( T \), to obtain that:

\[ f_0^T(Y^{\ast}_{in}, Y_n) \, dt + f_0^T(Y_n(t) \varphi_2(t) \, dt = f_0^T(f_1(Y^0_{1}, u_{in}) \varphi_2(t) + f_0^T(f_2(Y^0_{1}, u_{in}) \varphi_3(t) + f_0^T(f_3(Y^0_{1}, u_{in}) \varphi_4(t)), \] for all \( \varphi_i \in C[Q] \).

Using Lemma 1 for the \( 1^{st} \) terms in the L.H.S. of Eq.64 and Eq.65, they become resp.:

\[ f_0^T(Y^{\ast}_{in}, Y_n) \, dt + f_0^T(Y_n(t) \varphi_2(t) \, dt = f_0^T(f_1(Y^0_{1}, u_{in}) \varphi_2(t) + f_0^T(f_2(Y^0_{1}, u_{in}) \varphi_3(t) + f_0^T(f_3(Y^0_{1}, u_{in}) \varphi_4(t)), \] for all \( \varphi_i \in C[Q] \).
\[ \frac{1}{2} \| \tilde{y}(T) \|^2_{L^2(\Omega)} - \frac{1}{2} \| \tilde{y}(0) \|^2_{L^2(\Omega)} + \int_0^T \| \tilde{y} \|^2_{H^1(\Omega)} \, dt = - \int_0^T \left( (f_1(y_1, u_1), v_1) + (f_2(y_2, u_2), v_2) + (f_3(y_3, u_3), v_3) + (f_4(y_4, u_4), v_4) \right) dt. \]

Since \[ \frac{1}{2} \| \tilde{y}_n(T) - \tilde{y}(T) \|^2_{L^2(\Omega)} + \int_0^T (\tilde{y}_n - \tilde{y}, \tilde{y}_n - \tilde{y}) dt = A_1 - B_1 - C_1, \] where
\[
A_1 = \frac{1}{2} \| \tilde{y}_n(T) \|^2_{L^2(\Omega)} - \frac{1}{2} \| \tilde{y}_n(0) \|^2_{L^2(\Omega)} + \int_0^T \| \tilde{y}_n \|^2_{H^1(\Omega)} dt, \\
B_1 = \frac{1}{2} \left( \tilde{y}_n(T), \tilde{y}_n(T) \right) - \frac{1}{2} \left( \tilde{y}_n(0), \tilde{y}_n(0) \right) + \int_0^T \left( \tilde{y}_n(t), \tilde{y}_n(t) \right) dt, \\
and \\
C_1 = \frac{1}{2} \left( \tilde{y}(T), \tilde{y}(T) \right) - \frac{1}{2} \left( \tilde{y}(0), \tilde{y}(0) \right) + \int_0^T \left( \tilde{y}(t), \tilde{y}(t) \right) dt.
\]

Since \( \tilde{y}_n \to \tilde{y}_0 \) converges strongly in \( L^2(\Omega) \), \( \tilde{y}_n(T) \to \tilde{y}(T) \) converges strongly in \( L^2(\Omega) \), 70.

Then, from (69) and (70) we have
\[
\begin{align*}
& \left( \tilde{y}_n(T), \tilde{y}_n(0) - \tilde{y}(0) \right) \to 0, \\
& \left( \tilde{y}(T), \tilde{y}_n(T) - \tilde{y}(T) \right) \to 0,
\end{align*}
\]
and
\[
\begin{align*}
& \left( \tilde{y}_n(T) - \tilde{y}(0) \right) \| \mathbf{y} \|^2_{L^2(\Omega)} \to 0, \\
& \left( \| \tilde{y}_n(T) - \tilde{y}(T) \|^2_{L^2(\Omega)} \to 0. \\
\end{align*}
\]
Since \( \tilde{y}_n \to \tilde{y} \) converges weakly in \( L^2(I, V) \), then,
\[
\int_0^T \left( \tilde{y}_n(t), \tilde{y}_n(t) - \tilde{y}(t) \right) dt = 0, \]
Again since \( \tilde{y}_n \to \tilde{y} \) converges weakly in \( L^2(Q) \), from the continuity of integral
\[
\int_0^T \left( f_i(y_{in}, u_i), v_i \right) dt \text{ w.r.t. } y_i \text{ and } u_i \text{ and } \tilde{y}_n \to \tilde{y} \text{ converges strongly in } L^2(Q), \forall i = 1, 2, 3, 4.
\]
Now, when \( n \to \infty \) in both sides of Eq.68, one has the following results:
(1) The 1st two terms in the L.H.S. of Eq.68 are tending to zero from Eq.72.
(2) From Eq.64 and Eq.74, one has
\[
\text{Equation } \begin{align*}
& A_1 = \int_0^T \left( f_1(y_{1n}, u_1), v_{1n} \right) + (f_2(y_{2n}, u_2), v_{2n}) + (f_3(y_{3n}, u_3), v_{3n}) + (f_4(y_{4n}, u_4), v_{4n}) dt + \\
& \int_0^T \left( f_1(y_1, u_1), v_1 \right) + (f_2(y_2, u_2), v_2) + (f_3(y_3, u_3), v_3) + (f_4(y_4, u_4), v_4) dt.
\end{align*}
\]
(3) Equation \( B_1 \to \text{L.H.S.} \) of Eq.70
\[
\int_0^T \left( f_1(y_{1n}, u_1), v_1 \right) + (f_2(y_{2n}, u_2), v_2) + (f_3(y_{3n}, u_3), v_3) + (f_4(y_{4n}, u_4), v_4) dt + \\
\int_0^T \left( f_1(y_1, u_1), v_1 \right) + (f_2(y_2, u_2), v_2) + (f_3(y_3, u_3), v_3) + (f_4(y_4, u_4), v_4) dt.
\]
(4) The 1st two terms in Equation \( C_1 \) are tending to zero from Eq.71, and the last term also tends to zero from Eq.73, from these convergences and the results, Eq.68 gives:
\[
\int_0^T \| \tilde{y}_n - \tilde{y} \|^2_{H^1(\Omega)} dt = \int_0^T \| \tilde{y}_n - \tilde{y}, \tilde{y}_n - \tilde{y} \|^2 dt \text{ where } n \to \infty, \to 0, \text{ gives } \tilde{y}_n \to \tilde{y} \text{ converges strongly in } L^2(I, V).
\]

The uniqueness of the Solution:
Let \( \tilde{y} \) and \( \tilde{y}_2 \) are two QSVS of the WF of the QSVEs Eq.8, Eq.10, Eq.12, and Eq.14, then from Eq.8 one has:
\[
\begin{align*}
& (y_1, v_1) + (\forall y_1, v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1(y_1, u_1), v_1), \forall v_1 \in V, \\
& (y_1, v_1) + (\forall y_1, v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1(y_1, u_1), v_1), \forall v_1 \in V.
\end{align*}
\]
By subtracting the 2nd equation from the 1st one, then substituting \( v_1 = y_1 - y_1 \), one has that:
\[
\begin{align*}
& (y_1 - y_1, v_1) + (y_1 - y_1, v_1) - (y_2 - y_2) + (y_3 - y_3, y_1 - y_1) + (y_4 - y_4, y_1 - y_1) = (f_1(y_1, u_1) - f_1(y_1, u_1), y_1 - y_1), \\
& (y_2 - y_2) + (y_2 - y_2) - (y_3 - y_3, y_2 - y_2) + (y_4 - y_4, y_2 - y_2) = (f_2(y_2, u_2) - f_2(y_2, u_2), y_2 - y_2),
\end{align*}
\]
Same ways can use to get:
\[
\begin{align*}
& (y_1 - y_1, v_1) + (y_1 - y_1, v_1) - (y_2 - y_2) + (y_3 - y_3, y_1 - y_1) + (y_4 - y_4, y_1 - y_1) = (f_1(y_1, u_1) - f_1(y_1, u_1), y_1), \\
& (y_2 - y_2) + (y_2 - y_2) - (y_3 - y_3, y_2 - y_2) + (y_4 - y_4, y_2 - y_2) = (f_2(y_2, u_2) - f_2(y_2, u_2), y_2 - y_2),
\end{align*}
\]
Adding Eq.75 – Eq.78, using Lemma 1 for the 1st term of the obtained equations, the 2nd term of the L.H.S. of the obtained equation is non-negative, IBS w.r.t. \( t \) from 0 to \( t \), then using Assumptions A-II of the R.H.S. of it, and then by applying the CBGI to obtain that:
\[
\| (\tilde{y} - \tilde{y}) (t) \|^2_{L^2(\Omega)} = 0, \forall t \in I.
\]
Again, IBS of the obtained equation w.r.t. \( t \) from 0 to \( T \), using the given IC and the above result for the R.H.S. of the equation, one gets that:
\[
\begin{align*}
& \int_0^T \| \tilde{y} - \tilde{y} \|^2_{L^2(\Omega)} dt \leq L \int_0^T \| \tilde{y} - \tilde{y} \|^2_{L^2(\Omega)} dt + \\
& \int_0^T \| \tilde{y} - \tilde{y} \|^2_{H^1(\Omega)} dt \leq 0, \\
& \| \tilde{y} - \tilde{y} \|^2_{L^2(I, V)} = 0, \text{ which implies to } \tilde{y} = \tilde{y}.
\end{align*}
\]
Example 1: Let Ω = (0,1) × (0,1), I = [0,1], and the QNLBPBV is as given by Eq.1 – Eq.4, with
\[ f_i(x,t,y_i,u_i) = h_i(x,t) + Sin(y_i) + u_i - Sin(y_i) - u_i, \]
where \( h_i(x,t) \) be a given function, \( \forall i = 1,2,3,4. \)

With the boundary and the initial conditions Eq.5 and Eq.6 are given as:
\[ y_1(x,t) = 0, \quad \text{on } \Sigma, \quad y_1^0(x) = 4(x_1 x_2 - x_1^2 x_2 - x_2^2 x_2^2), \quad \text{on } \Omega, \quad \forall i = 1,2,3,4. \]

Since the function \( f_i \) satisfies all the assumptions A, for each \( i = 1,2,3,4 \), let \( \bar{u} \in L^2(\Omega) \) be any given QCCCV, then from Thereon 1, the WF Eq.8 – Eq.15, has a unique QSVS \( \bar{y} \).

Existence of a Quaternary Classical Continuous Optimal Control:
In this section, the following Theorems and Lemmas will be needed later in the study of the existence of the QCCOCV.

Theorem 2:

a) In addition to Assumptions A, consider that \( \tilde{y} \) and \( \tilde{y} + \delta \tilde{y} \) are the QSVs corresponding to the bounded QCCCVs in \((L^2(Q))^4\), \( \bar{u} \) and \( \bar{u} + \delta \bar{u} \) resp., then
\[ \|\delta \tilde{y}\|_{L^2(\Omega)} \leq M\|\delta \bar{u}\|_{L^2(\Omega)}, \]
where \( M \) is a constant.

b) With Assumptions A, the operator \( \bar{u} \rightarrow \tilde{y} \bar{u} \) from \( L^2(\Omega) \) into \( L^2(\Omega) \) is continuous.

Proof: Let \( \bar{u} \in L^2(\Omega) \), then by Theorem 1 there exist \( \bar{y} \) and \( \bar{y} \) which are their corresponding QSVs and are satisfied Eq.8 – Eq.15 i.e.:
\[
\begin{align*}
(\bar{y}_1, v_1) + (\bar{y}_2, v_1) + (\bar{y}_3, v_1) + (\bar{y}_4, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 79 \\
(\bar{y}_1, v_1) + (\bar{y}_2, v_1) + (\bar{y}_3, v_1) + (\bar{y}_4, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 80 \\
(\bar{y}_2, v_1) + (\bar{y}_2, v_1) + (\bar{y}_2, v_1) + (\bar{y}_2, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 81 \\
(\bar{y}_2, v_1) + (\bar{y}_2, v_1) + (\bar{y}_2, v_1) + (\bar{y}_2, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 82 \\
(\bar{y}_3, v_1) + (\bar{y}_3, v_1) + (\bar{y}_3, v_1) + (\bar{y}_3, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 83 \\
(\bar{y}_4, v_1) + (\bar{y}_4, v_1) + (\bar{y}_4, v_1) + (\bar{y}_4, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 84 \\
(\bar{y}_4, v_1) + (\bar{y}_4, v_1) + (\bar{y}_4, v_1) + (\bar{y}_4, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 85 \\
(\bar{y}_3, v_1) + (\bar{y}_3, v_1) + (\bar{y}_3, v_1) + (\bar{y}_3, v_1) &= (f_1(\bar{y}_1, \bar{u}), v_1), \quad 86 \\
\end{align*}

By subtracting Eq.8 – Eq.15 from Eq.79 – Eq.86 resp. and setting \( y_1 = \tilde{y} - \bar{y}, \delta \bar{u} = \bar{u} - u_i, \) \( \forall i = 1,2,3,4. \)

Hence, for \( \delta y_2, \delta y_3, \delta y_4 \), the obtained equations give that:
\[ (\delta y_{2t}, v_2) + (\nabla \delta y_{2}, \nabla v_2) + (\delta y_{3}, v_2) + (\delta y_{4}, v_2) = (f_2(y_2 + \delta y_{2}, u_2 + \delta u_2), v_2), \]
where \( h_i(x,t) \) be a given function, \( \forall i = 1,2,3,4 \).
Now since \( \|\tilde{\delta}y\|^2_{L^2(Q)} \leq \max_{t \in [0,T]} \|\tilde{\delta}y(t)\|^2_{L^2(\Omega)} t^2 dt \leq M^2 \|\tilde{\delta}u\|^2_{L^2(Q)} \).

Then, \( \|\tilde{\delta}y\|^2_{L^2(Q)} \leq M \|\tilde{\delta}u\|^2_{L^2(Q)} \), where \( M^2 = TM^2 \).

Using the same way used in the above steps for the R.H.S. of Eq.95 with \( t = T \), one has that:

\[
\|\tilde{\delta}y(T)\|^2_{L^2(Q)} + 2 \int_0^T \|\tilde{\delta}y\|^2_{\mathcal{C}^1(\Omega)} dt \leq \bar{L}_1 \|\tilde{\delta}u\|^2_{L^2(Q)} + \bar{L}_2 \|\tilde{\delta}y\|^2_{L^2(Q)},
\]

\[
\|\tilde{\delta}y\|^2_{L^2(L^2)} \leq M^2 \|\tilde{\delta}u\|^2_{L^2(Q)},
\]

where \( M^2 = (\bar{L}_1 + \bar{L}_2 M^2)/2 \), and

\[
\|\tilde{\delta}y\|^2_{L^2(L^2)} \leq M \|\tilde{\delta}u\|^2_{L^2(Q)},
\]

where \( M \) denotes the various constants.

b) From the results of part (a) above, easily one can get that the operator \( \tilde{u} \mapsto \tilde{y} \) is Lipschitz continuous.

**Lemma 3:**
With Assumption B, the functional \( \tilde{u} \mapsto G_0(\tilde{u}) \) is continuous on \( L^2(\Omega) \).

**Proof:** By applying Assumption B and Proposition 1, the integral \( \int_0^T \int \bar{g}_0(x,t,y_i,u_i) \, dx \, dt \) is continuous on \( L^2(Q) \), \( \forall i = 1,2,3,4 \), hence \( G_0(\tilde{u}) \) is continuous on \( L^2(Q) \).

**Theorem 3:**
Consider the control set is of the form \( \tilde{W} = W_1 \times W_2 \times W_3 \times W_4 \) with \( \tilde{U} \) is convex and compact, \( \tilde{W}_A \neq \emptyset \), and \( f_i \). (\( \forall i = 1,2,3,4 \)) has the form:

\[
f_i(x,t,y_i,u_i) = f_1(x,t,y_i) + f_2(x,t,u_i),
\]

with \( f_2(x,t,y_i) \leq \eta_i(x,t) + c_i |y_i| \) and \( f_2(x,t,y_i) \leq k_i \), with \( \eta_i \in L^2(Q) \), \( k_i, c_i \geq 0 \).

If \( g_{0i} \) for each \( i = 1,2,3,4 \) is convex w.r.t. \( u_i \) for each fixed \( (x,t,y_i) \). Then there exists a QCCOVCV for the considered CCCOP.

**Proof:** From the Assumptions on \( U_i \subset \mathbb{R} \), \( \forall i = 1,2,3,4 \) and the Egorov’s theorem, one gets that \( \tilde{W} \) is weakly compact. Since \( \tilde{W}_A \neq \emptyset \), so there is \( \tilde{u} \in \tilde{W}_A \) and there is a minimum sequence \( \{ \tilde{u}_k \} \) with \( \tilde{u}_k \in \tilde{W}_A \) s.t. \( \lim_{k \to \infty} G_0(\tilde{u}_k) = \inf_{\tilde{u} \in \tilde{W}_A} G_0(\tilde{u}) \).

Since \( \tilde{u}_k \in \tilde{W}_A \), \( \forall k \) but \( \tilde{W} \) is, there is a subsequence of \( \{ \tilde{u}_k \} \) which converges weakly to some \( \tilde{u} \in \tilde{W} \), i.e. \( \tilde{u}_k \rightharpoonup \tilde{u} \) WK in \( L^2(Q) \), with \( \|\tilde{u}_k\|_Q \leq c \), \( \forall k \). From Theorem 1 for each QCCOVCV \( \tilde{u}_k \), the weak form of the QVSEs \( \tilde{y}_k \) converges weakly to some \( \tilde{y} \) w.r.t. these norms, i.e.:

\[
\tilde{y}_k \rightharpoonup \tilde{y} \text{ converges weakly in the spaces } L^\infty(I,L^2(\Omega)), L^2(\Omega), \text{ and in } L^2(I,L^2(\Omega)).
\]

Now, to show that the norm \( \|\tilde{y}_k\|_{L^2(I,L^2)} \) is bounded:

The weak form Eq.17 – Eq.24 can be rewritten as:

\[
\langle y_{1k}, v_1 \rangle = \langle \nabla y_{1k}, \nabla v_1 \rangle - (y_{1k}, v_1) + \langle y_{2k}, v_1 \rangle - (y_{3k}, v_1) - (y_{4k}, v_1) + (f_1(y_{1k},u_{1k}), v_1),
\]

\[
\langle y_{2k}, v_2 \rangle = \langle \nabla y_{2k}, \nabla v_2 \rangle - (y_{1k}, v_2) - (y_{2k}, v_2) + \langle y_{3k}, v_2 \rangle + (f_2(y_{2k},u_{2k}), v_2).
\]

\[
\langle y_{3k}, v_3 \rangle = \langle \nabla y_{3k}, \nabla v_3 \rangle + (y_{1k}, v_3) - (y_{2k}, v_3) - (y_{3k}, v_3) + (f_2(y_{3k},u_{3k}), v_3),
\]

\[
\langle y_{4k}, v_4 \rangle = \langle \nabla y_{4k}, \nabla v_4 \rangle + (y_{1k}, v_4) - (y_{2k}, v_4) + (y_{3k}, v_4) - (y_{4k}, v_4) + (f_4(y_{4k},u_{4k}), v_4).
\]

By adding the above equalities, integrating both sides from 0 to \( T \), then taking the absolute value, using the Cauchy-Schwartz inequality, and finally using Assumptions A-i, it yields:

\[
\int_0^T \langle \tilde{y}_k, \tilde{v} \rangle dt \leq \langle \nabla y_{1k}, \tilde{v} \rangle + \langle y_{1k}, \tilde{v} \rangle + \langle y_{2k}, \tilde{v} \rangle + \langle y_{3k}, \tilde{v} \rangle + \langle y_{4k}, \tilde{v} \rangle + \langle f_1(y_{1k},u_{1k}), \tilde{v} \rangle + \langle f_2(y_{2k},u_{2k}), \tilde{v} \rangle + \langle f_2(y_{3k},u_{3k}), \tilde{v} \rangle + \langle f_4(y_{4k},u_{4k}), \tilde{v} \rangle.
\]

Since for each \( i = 1,2,3,4 \), the following are held:

\[
\|\nabla y_{ik}\|_{L^2} \leq \|\nabla i\|_{L^2}, \quad \|\nabla y_{ik}\|_{L^2} \leq \|\tilde{v}\|_{L^2}, \quad \|\tilde{v}\|_{L^2} \leq \|\tilde{v}\|_{L^2}, \quad \|\tilde{v}\|_{L^2} \leq \|\tilde{v}\|_{L^2}, \quad \|\tilde{v}\|_{L^2} \leq \|\tilde{v}\|_{L^2}, \quad \|\tilde{v}\|_{L^2} \leq \|\tilde{v}\|_{L^2}.
\]

Then, the above inequality leads to:

\[
\int_0^T \langle \tilde{y}_k, \tilde{v} \rangle dt \leq 4\|\nabla y_{ik}\|_{L^2} \|\nabla v\|_{L^2} + 16\|\tilde{v}\|_{L^2} \|\tilde{v}\|_{L^2} + \langle b(c) \tilde{c}, \tilde{v} \rangle,
\]

where \( b(c) = \tilde{b}_5(c) + \tilde{b}_6(c) + \tilde{b}_7(c) + \tilde{b}_8(c), \quad \tilde{b}_5(c) = \tilde{b}_1 + c_1 \tilde{b}_3(c) + c_1 \tilde{c}_1, \quad \tilde{b}_6(c) = \tilde{b}_2 + c_2 \tilde{b}_3(c) + \tilde{c}_2 \tilde{c}_2, \quad \tilde{b}_7(c) = \tilde{b}_3 + c_3 \tilde{b}_3(c) + \tilde{c}_3 \tilde{c}_3 \).
\[ \bar{b}_k(c) = b_k + c_4 b_4(c) + c_4 \bar{c}_4, \] setting \( \bar{b}(c) = 20 b_2(c) + \bar{b}(c), \)

then

\[ ||\tilde{y}_{kt}||_{L^2(I,V')} = sup \left\{ \int_0^T |(\tilde{y}_{kt}, \tilde{v}) dt | \bigg| |\tilde{b}(c)| \right\}, \]

thus

\[ ||\tilde{y}_{kt}||_{L^2(I,V')} \leq \bar{b}(c), \forall \tilde{y}_{kt} \in \tilde{V}'. \]

Eq.42 also is held here and gives the injections

\[ L^2(I, V) \subseteq L^2(I, V') \] and \( (L^2(I, V'))' \subseteq L^2(I, V') \) are continuous but the injection \( L^2(I, V) \subseteq L^2(I, V') \) is compact. Then by compactness theorem, there is a subsequence of \( \{\tilde{y}_k\} \) say again \( \{\tilde{y}_k\} \) s.t. \( \tilde{y}_k \rightarrow \tilde{y} \) converges strongly in \( L^2(I, V') \).

Now, since \( \mathfrak{V}_k, y_{ik} (i = 1, 2, 3, 4) \) is the QSVS of Eq.17 – Eq.24 corresponding to the QCCCV \( u_{ik}, i.e., \)

\[ \{ y_{ik}, v_1 \} + (\nabla y_{ik}, v_1) + (y_{ik}, v_1) - (y_{ik}, v_1) + (y_{ik}, v_3), \]

\[ (f_{11}(x, t), v_1, t) + f_{12}(x, t), u_{ik}, v_1, t), \]

\[ (f_{22}(x, t), u_{ik}, v_2, t), \]

\[ (f_{32}(x, t), u_{ik}, v_2, t), \]

\[ (f_{31}(x, t), y_{ik}, v_3, t), \]

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\[ (f_{31}(x, t), y_{ik}, v_3, t), \]

\[ (f_{31}(x, t), y_{ik}, v_3, t), \]

\[ (f_{31}(x, t), y_{ik}, v_3, t), \]
This Eq. 113 – Eq. 116 are held for each \( v_i \in C[\bar{I}], \) but \( C[\bar{I}] \) is dense in \( V \), then they are held for every \( v_i \in V, \forall i = 1, 2, 3, 4 \).

Now, we have following two cases:

**Case 1:** Choose, \( (\forall i = 1, 2, 3, 4), \) \( v_i \in D[0, T] \), i.e. \( \phi_i(0) = \phi_i(T) = 0 \) then by using integration by parts for the \( 1^{st} \) terms in the L.H.S. of each one of Eq.73 – Eq.76, one gets \( \forall v_i \in V, \forall i = 1, 2, 3, 4 \) that:

\[
\begin{align*}
&\int_0^T (\dot{y}_t, v_1) \phi_1(t) dt + \int_0^T (\nabla y_t, \nabla v_1) + \\
&(y_1, v_1) \phi_1(t) dt - \int_0^T (\dot{y}_t, v_2) \phi_1(t) dt + \\
&\int_0^T (y_2, v_1) \phi_1(t) dt + \int_0^T (y_3, v_1) \phi_1(t) dt = \\
&\int_0^T (f_{11}(x, t, y_1), v_1) \phi_1(t) dt + \\
&\int_0^T (f_{12}(x, t), u_1, v_1) \phi_1(t) dt + \\
&\int_0^T (f_{13}(x, t, u_1, v_1) \phi_1(t) dt \quad \forall v_i \in C[\bar{I}], \quad 109
\end{align*}
\]

Finally, using (Eq.104, Eq.108 for \( i = 1 \), Eq.109 & Eq.110) in Eq.100, using (Eq.105, Eq.108 (for \( i = 2 \), Eq.111 (for \( i = 2 \)) & Eq.112(for \( i = 2 \)) in Eq.101, also Eq.109, Eq.110, Eq.111&Eq.112(for \( i = 3 \)) in Eq.102 and finally (Eq.107, Eq.108(for \( i = 4 \), Eq.111 & Eq.112(for \( i = 4 \) in Eq.103) to get that:

\[
\begin{align*}
&\int_0^T (y_1, v_1) \phi_1(t) dt + \int_0^T (\nabla y_1, \nabla v_1) + \\
&(y_1, v_1) \phi_1(t) dt - \int_0^T (\dot{y}_t, v_2) \phi_1(t) dt + \\
&\int_0^T (y_3, v_1) \phi_1(t) dt + \int_0^T (y_4, v_1) \phi_1(t) dt = \\
&\int_0^T (f_{11}(x, t, y_1), v_1) \phi_1(t) dt + \\
&\int_0^T (f_{12}(x, t), u_1, v_1) \phi_1(t) dt + \\
&\int_0^T (f_{13}(x, t, u_1, v_1) \phi_1(t) dt \quad \forall v_i \in C[\bar{I}], \quad 113
\end{align*}
\]

Thus Eq.113 – Eq.116 are held for each \( v_i \in C[\bar{I}], \) but \( C[\bar{I}] \) is dense in \( V \), then they are held for every \( v_i \in V, \forall i = 1, 2, 3, 4 \).

Case 2: Choose, \( (\forall i = 1, 2, 3, 4), \) \( v_i \in C^1[\bar{I}], \) s.t. \( \phi_i(T) = 0 \) and \( \phi_i(0) \neq 0, \) then using integration by
parts for 1st terms in the L.H.S. of Eq.113 –
Eq.116, one has:
\[
- \int_{T}^{T}(y_1, v_1)_{\partial x} dt + \int_{T}^{T}[(\nabla y_1, \nabla v_1) + (y_1, v_1)]_{\partial x} dt - \int_{T}^{T}(y_2, v_1)_{\partial x} dt + 
\]
\[
\int_{T}^{T}(y_3, v_1)_{\partial x} dt + \int_{T}^{T}(y_4, v_1)_{\partial x} dt = \int_{T}^{T}(f_{11}(x, t, y_1), v_1)_{\partial x} dt + 
\]
\[
\int_{T}^{T}(f_{12}(x, t, y_1, v_1)_{\partial x} dt) + (y_1(0), v_1)_{\partial x} dt, 
\]
\[
\lim_{k \to \infty} \int_{Q}^{Q}(g_{01}(x, t, y_i, u_{ik}) - g_{01}(x, t, v_{ik}, u_{ik})) dx dt + \lim_{k \to \infty} \int_{Q}^{Q} g_{01}(x, t, y_{ik}, u_{ik}) dx dt 
\]
Then, by Eq.125:
\[
\lim_{k \to \infty} \int_{Q}^{Q} g_{01}(x, t, y_i, u_i) dx dt \leq 
\]
\[
\lim_{k \to \infty} \int_{Q}^{Q} g_{01}(x, t, y_{ik}, u_{ik}) dx dt ,
\]
i.e. \(G_0(\tilde{u})\) is W.L.S.C. w.r.t. \(\tilde{y}, \tilde{u}\)
but \(G_0(\tilde{u}) \leq \lim_{k \to \infty} G_0(\tilde{u}_k) = \lim_{k \to \infty} G_0(\tilde{u}_k) = 
\]
\[
\inf_{\tilde{u} \in \tilde{W}_A} G_0(\tilde{u}_k), \text{ therefore } G_0(\tilde{u}) = \min_{\tilde{u} \in \tilde{W}_A} G_0(\tilde{u}_k).
\]
Then, \(\tilde{u}\) is a QCCOCV.  

**Example 2:** Consider the classical continuous optimal control problem, consisting of the quaternary nonlinear parabolic boundary value problem which is given in Example 1(above), with \(f_{11}(x, t, y_i) = h_1(x, t) - k_1(x, t)\tilde{u}_i + \sin y_i - \sin y_i \)
\[f_{12}(x, t) = k_i(x, t)\tilde{u}_i, \]
where \(h_i(x, t)\) and \(k_i(x, t)\) are given functions, \(\forall i = 1, 2, 3, 4.\)
The cost function is \(G_0(\tilde{u}) = \sum_{i=1}^{4} \int_{Q}(y_i - \tilde{y}_i)^2 + \)
\[(u_i - \tilde{u}_i)^2) dx dt, \text{ with } \tilde{U} = [-1, 1]^4.\]
Since for each \(i = 1, 2, 3, 4, \) \(g_{01}(x, t, y_i, u_i) =
\[(y_i - \tilde{y}_i)^2 + (u_i - \tilde{u}_i)^2,\]
satisfies the assumption B, and \(f_{11}(x, t, y_i), f_{12}(x, t)\) satisfy all the hypotheses in theorem 3, then there is a quaternary classical continuous optimal control vector.

**Conclusions and Discussions:**
In this work, the continuous classical optimal control dominated by a quaternary nonlinear parabolic boundary value problem has been studied. Under suitable assumptions, the existence and uniqueness theorem of the quaternary state vector solution for the weak form of the quaternary nonlinear parabolic boundary value problem with a given quaternary classical continuous control vector have been stated and proved via the Galerkin Method, and the first compactness theorem. In addition to the continuity operator between the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem and the corresponding quaternary classical continuous control vector have been proved. The existence of a quaternary classical continuous optimal control vector dominated by the considered quaternary nonlinear parabolic boundary value problem has been stated and proved under suitable assumptions.
The study of the proposed problem is very interesting in the field of applied mathematics since the proposed problem represents a generalization for a heat equation; furthermore, it represents multi objectives problems that have many applications. Also, these results are very important because they
give the green light about the ability of solving such problems numerically. This point serves as a future work for the topic.

Authors' declaration:
- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

Authors' contributions statement:
J. A. A. Al. and W. A. A. Al. contributed to the design and implementation of the research, the proof of the theorems, and the writing of the manuscript.

References:
السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة

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الخلاصة:

في هذا العمل هدفنا هو دراسة السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة، بوجود شروط مناسبة. تم ذكر نص وبرهان مبرهنة وجود ووحدانية الحل لمنحجه الحالة الرباعي المستمر للصيغة الضعيفة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة عندما يكون متجه السيطرة التقليدية المستمرة معلوماً، بواسطة طريقة كاليكركن والبرهنة المرصوصة الأولى. تم برهان عامل الاستمرارية بين متجه الحالة الرباعي المستمر للصيغة الضعيفة لمسالة القيم الحدودية الرباعية المكافئة ومنحجه السيطرة التقليدية المستمرة. أيضاً تم برهان مبرهنة وجود متجه رباعي لسيطرة أمثلية تقليدية مستمرة لهذه المسألة بوجود شروط مناسبة.

الكلمات المفتاحية: السيطرة الامثلية التقليدية، دالة الهدف، طريقة كاليكركن، استمرارية ليبشتز، مسائل القيم الحدودية المكافئة.