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The Classical Continuous Optimal Control for Quaternary Nonlinear Parabolic System

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Abstract:

In this paper, our purpose has been to study the classical continuous optimal control problem for the quaternary nonlinear parabolic boundary value problem. Under suitable assumptions and with the given quaternary classical continuous control vector, the existence and uniqueness theorem for the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem has been stated and proved via the Galerkin Method, and the first compactness theorem. Furthermore, the continuity operator between the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem and the corresponding quaternary classical continuous control vector have been proved. The existence of a quaternary classical continuous optimal control vector has been stated and proved under suitable conditions.

Keywords: Classical Optimal Control, Cost Function, Galerkin Method, Lipschitz continuity, Parabolic Boundary Value Problems.

Introduction:

Optimal control problem (OCP) means endogenously controlling a parameter in a mathematical model to produce an optimal (cost) output. The problem comprises an objective (or cost) function, which is a function of the state and control variables, and the constraints on the control. The problem seeks to optimize the objective function subject to the constraints construed by the model describing the evolution of the underlying system[1]. Optimal control problems (OCPs) play an important role in many practical applications, such as in weather conditions[2], economics[3], robotics[4], aircraft[5], medicine[6], and many other scientific fields. They are two types of OCPs; the classical and the relax type, each one of these two types is dominated either by nonlinear ODEs[7] or by nonlinear PDEs (NLPDEs)[8]. The classical continuous optimal control problem (CCOCP) dominated by nonlinear parabolic or elliptic or hyperbolic PDEs are studied in[9-11] respectively (resp.). Later, the study of the CCOCPs dominated by the three types of nonlinear PDEs is generalized in [12-14] to deal with CCOCPs dominating by coupling NLPDEs of these types resp., and then these studies are generalized also to deal with CCOCPs dominated by triple and NLPDEs of the three types [15-17].

In each type of these CCOCPs the problem consists of; an initial or a boundary value problem (the dominating equations), the objective (cost) function of the classical continuous control vector, the state vector, and the constraints on the control vector. The study in each one of these problems included; the state and proves the existence of a unique “ state “ vector solution for the weak form (WF) which is usually obtained from the initial or the boundary value problem (the dominating equations) where the classical continuous control is known. The continuity of the Lipschitz operator between the state vector solution of the WF for the dominating equations and the corresponding classical continuous control vector is proved. The existence theorem of a classical continuous optimal control vector for the problem is stated and proved under suitable conditions. All of the above-mentioned studies encouraged us to think about generalizing the study of the CCOCP dominated by triple NLPDEs of parabolic type to a CCOCP dominated by quaternary nonlinear parabolic boundary value problem (QNLPBVP). According to this idea for the generalization, the mathematical model for the dominating equation is needed to be found, as well as the cost function, the spaces of definition for the control, and the state vectors, which all of them are needed to be generalized. Hence all the theorems and the results which were accompanied in the above studies for the CCOCPs dominated by the couple and triple nonlinear PDEs of the three types are needed to be stated and proved, for the "new" proposed CCOCP which is dominating by the QNLPPDEs.

The study of the proposed CCOCP is very interesting in the field of applied mathematics because the proposed model represents a generalization of the heat equation from one side, and the other, it represents a multi objectives problem that has many applications. Furthermore, the results of this paper are very useful, because give the green light about the ability for solving the problem numerically.

The study of the CCOCP dominated by the QNLPBVP which is proposed in this paper starts with the state and proof of the existence theorem of the quaternary state vector solution (QSVS) of the weak form (WF) for the QNLPBVP using the Galerkin Method (GM) with the first compactness theorem, under suitable conditions and when the quaternary classical continuous control vector (QCCCV) is known. The continuity of the Lipschitz operator between the state vector solution of the WF for the QNLPBVP and the corresponding classical continuous control vector are proved. The existence theorem of a quaternary classical continuous optimal control vector (QCCOCV) is stated and demonstrated under suitable conditions.

Problem Description:

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and regular region with Lipschitz boundary $\Gamma = \partial\Omega$, $x = (x_1, x_2)$, $Q = I \times \Omega$, $I = [0, T]$, $\Gamma = \partial\Omega$, $\Sigma = \Gamma \times$ I. The CCOCP consists of the following QNLPBVP, which are constructed by us, are given by (with $y_i = y_i(x,t)$ and $u_i = u_i(x,t), \forall\, i = 1,2,3,4$):

$y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x,t,y_1,u_1)$, in Q 1

$y_{2t} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x,t,y_2,u_2)$, in Q 2

$y_{3t} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x,t,y_3,u_3)$, in Q 3

$y_{4t} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x,t,y_4,u_4)$, in Q 4

With the following boundary conditions (BCs) and initial conditions (ICs):

$y_i(x,t) = 0$, $\forall\, i = 1,2,3,4.$ on Σ 5

$y_i(x,0) = y_i^0(x)$, $\forall\, i = 1,2,3,4.$ on Ω 6

Where $\vec{y} = (y_1, y_2, y_3, y_4) \in \left(\mathcal{H}^2(\overline{\mathrm{Q}})\right)^4$ is the QSV, $\vec{u} = (u_1, u_2, u_3, u_4) \in \left(L^2(\mathrm{Q})\right)^4$ is the QCCCV and $(f_1, f_2, f_3, f_4) \in \left(L^2(\mathrm{Q})\right)^4$ is given, for all $x \in \Omega$.

The Set of Admissible Control is:

$\vec{W}_A = \left\{\vec{w} \in \left(L^2(\mathrm{Q})\right)^4 \middle| \vec{w} \in \vec{U} \subset \mathbb{R}^4 \text{ a.e. in Q}\right\}$.

The Cost Function is:

$G_0(\vec{u}) = \int_Q g_{01}(x,t,y_1,u_1)dxdt + \int_Q g_{02}(x,t,y_2,u_2)dxdt + \int_Q g_{03}(x,t,y_3,u_3)dxdt + \int_Q g_{04}(x,t,y_4,u_4)dxdt.$ 7

Where $\vec{y} = \vec{y}_{\vec{u}}$ is the QSVS of Eq.1–Eq.6 corresponding to the QCCCV $\vec{u}$.

The classical continuous optimal control problem is to find $\vec{u} \in \vec{W}_A$, s.t. :

$G_0(\vec{u}) = \min_{\vec{w} \in \vec{W}_A} G_0(\vec{w})$.

The notations (v, v) and $\|v\|_{L^2(\Omega)}$ are referred to as the inner product and the norm in $L^2(\Omega)$, the notations $(v,v)_{\mathcal{H}_0^1(\Omega)}$ and $\|v\|_{\mathcal{H}^1(\Omega)}$ the inner product and the norm in $\mathcal{H}^1(\Omega)$, the $(\vec{v}, \vec{v})$ and $\|\vec{v}\|_{\boldsymbol{L^2(\Omega)}}$ the inner product and the norm in $\boldsymbol{L^2(\Omega)} = (L^2(\Omega))^4$ and $(\vec{v}, \vec{v}) = (v_1, v_1) + (v_2, v_2) + (v_3, v_3) + (v_4, v_4)$, $\|\vec{v}\|^2_{\boldsymbol{\mathcal{H}^1(\Omega)}} = \|v_1\|^2_{\mathcal{H}^1(\Omega)} + \|v_2\|^2_{\mathcal{H}^1(\Omega)} + \|v_3\|^2_{\mathcal{H}^1(\Omega)} + \|v_4\|^2_{\mathcal{H}^1(\Omega)}$ the inner product and the norm in $\vec{V} = \boldsymbol{\mathcal{H}_0^1(\Omega)} = \left(\mathcal{H}_0^1(\Omega)\right)^4$, $\vec{V}^*$ is the dual of $\vec{V}$, $\boldsymbol{L^2(I,V)} = \left(L^2(I,V)\right)^4$, $\boldsymbol{L^2(I,V^*)} = \left(L^2(I,V^*)\right)^4$, $\boldsymbol{L^2(Q)} = (L^2(\Omega))^4$, and $\left(\boldsymbol{L^2(Q)}\right)^*$ it is dual.

The Weak Form of the Quaternary State Vector Equations:

The WF of Eq.1 – Eq.6(when $\vec{y} \in \boldsymbol{\mathcal{H}_0^1(\Omega)}$, and $\forall v_i \in V_i = \mathcal{H}_0^1(\Omega), i = 1,2,3,4$) is

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1(y_1,u_1), v_1),$ 8

$(y_1^0, v_1) = (y_1(0), v_1),$ 9

$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2(y_2,u_2), v_2),$ 10

$(y_2^0, v_2) = (y_2(0), v_2),$ 11

$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3(y_3,u_3), v_3),$ 12

$(y_3^0, v_3) = (y_3(0), v_3),$ 13

$\langle y_{4t}, v_4\rangle + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4(y_4, u_4), v_4),$ 14

$(y_4^0, v_4) = (y_4(0), v_4),$ 15

The following assumptions, propositions, and Lemmas are important in our study of the QCCOC.

Assumptions A: Suppose for $(x,t) \in Q$, and $y_i, \bar{y}_i, u_i \in \mathbb{R}$, $\forall i = 1,2,3,4$ that:

(i) f_i is Carathéodory type (Cara. T.) on $Q \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and the following conditions w.r.t. y_i & u_i are held: $|f_i(x,t,y_i,u_i)| \le \eta_i(x,t) + c_i|y_i| + \acute{c}_i|u_i|$, where $\eta_i \in L^2(Q,\mathbb{R})$, and $c_i, \acute{c}_i > 0$.

(ii) f_i satisfies Lipschitz condition w.r.t. y_i, i.e.: $|f_i(x,t,y_i,u_i) - f_i(x,t,\bar{y}_i,u_i)| \le L_i|y_i - \bar{y}_i|$, $L_i > 0$.

Proposition 1 [18]**:**

Let $f: \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is Cara. T., let F be a functional, s.t. $F(y) = \int_\Omega f\big(x, y(x)\big)dx$, where Ω is a measurable subset of $\mathbb{R}^n$, and suppose that $\|f(x,y)\| \le \zeta(x) + \eta(x)\|y\|^\alpha$, $\forall (x,y) \in \Omega \times \mathbb{R}^n$, $y \in L^p(\Omega \times \mathbb{R}^n)$. Where $\zeta \in L^1(\Omega \times \mathbb{R})$, $\in L^{\frac{p}{p-\alpha}}(\Omega \times \mathbb{R})$, and $\alpha \in [0,p]$, if $p \in [1,\infty)$, and $\eta \equiv 0$, if $p = \infty$. Then, F is continuous on $L^p(\Omega \times \mathbb{R}^n)$.

Lemma 1 [19]**:**

Let V, H, V^* be three Hilbert spaces, each space included in the following one as in the following equality,

$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p\, dx\right)^{\frac{1}{p}}$ or $\|u\|_{L^\infty(\Omega)} = ess.\sup|u(x)|$.

V^* being the dual of V. If a function u belongs to $L^2(0,T;V)$ and it's derivative $\acute{u}$ belongs to $L^2(0,T;V^*)$, then u is almost everywhere (a.e.) equal to a function continuous from $[0,T]$ in to H and one have the following equality, which holds in the scalar distribution sense on $(0,T)$: $\frac{d}{dt}|u|^2 = 2\langle \acute{u}, u\rangle$.

Assumption B:

Consider g_{0i} (for each $i = 1,2,3,4$) is of Cara. T. on $Q \times \mathbb{R}^4$ and the following conditions w.r.t. y_i , u_i, hold: $|g_{0i}(x,t,y_i,u_i)| \le \eta_{0i}(x,t) + c_{0i1}(y_i)^2 + c_{0i2}(u_i)^2$. Where $y_i, u_i \in \mathbb{R}$ and $\eta_{0i} \in L^1(Q)$.

Lemma 2 [13]**:**

Let $g: \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is of Cara. T. on $\Omega \times \mathbb{R}^4$, and satisfies: $|g(x,y,u)| \le \eta(x) + cy^2 + \acute{c}u^2$, where $\eta \in L^1(\Omega,\mathbb{R})$, $c \ge 0$ and $\acute{c} \ge 0$.

Then, $\int_\Omega g(x,y,u)\, dx$ is continuous on $L^2(\Omega, \mathbb{R}^2)$, with $u \in U$, $U \subset \mathbb{R}$ is compact.

Main Results:

The Solution of the Weak Form:

This section deals with the state and proof of the existence and uniqueness theorem of QSVS for the weak form of the QNLPBVP under the suitable assumptions when the QCCCV is given.

Theorem 1: (The Existence and Uniqueness Theorem for the Weak Form)

With Assumptions A, for each given QCCCV $\vec{u} \in \boldsymbol{L^2(Q)}$, the weak form of Eq.8 – Eq.15 has a unique QSVS $\vec{y} \in \boldsymbol{L^2(I,V)}$, with $\vec{y}_t \in \boldsymbol{L^2(I,V^*)}$.

Proof: Let for any n, $\vec{V}_n = V_n \times V_n \times V_n \times V_n = V_{1n} \times V_{2n} \times V_{3n} \times V_{4n} \subset \vec{V}$ (i.e. each subspace V_{in} has its different basis) be the set of piecewise affine functions in Ω, let $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n})$ with $v_{in} \in \mathrm{V}_n (\forall i = 1,2,3,4)$ and $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$, let $\vec{y}_n$ be an approximation for $\vec{y}$ s.t.:

$y_{in} = \sum_{j=1}^n c_{ij}(t) v_{ij}(x), \quad \forall i = 1,2,3,4.$ 16

Where $c_{ij}(t)$ is an unknown function of t, $\forall i = 1,2,3,4$, $j = 1,2,\dots\dots,n$.

Hence, Eq.8 – Eq.15 become $\forall v_i \in V_n, i = 1,2,3,4$):

$\langle y_{1nt}, v_1\rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (y_{3n}, v_1) + (y_{4n}, v_1) = (f_1(y_{1n}, u_1), v_1),$ 17

$(y_{1n}^0, v_1) = (y_1^0, v_1),$ 18

$\langle y_{2nt}, v_2\rangle + (\nabla y_{2n}, \nabla v_2) + (y_{1n}, v_2) + (y_{2n}, v_2) - (y_{3n}, v_2) - (y_{4n}, v_2) = (f_2(y_{2n}, u_2), v_2),$ 19

$(y_{2n}^0, v_2) = (y_2^0, v_2),$ 20

$\langle y_{3nt}, v_3\rangle + (\nabla y_{3n}, \nabla v_3) - (y_{1n}, v_3) + (y_{2n}, v_3) + (y_{3n}, v_3) + (y_{4n}, v_3) = (f_3(y_{3n}, u_3), v_3),$ 21

$(y_{3n}^0, v_1) = (y_3^0, v_3),$ 22

$\langle y_{4nt}, v_4\rangle + (\nabla y_{4n}, \nabla v_4) - (y_{1n}, v_4) + (y_{2n}, v_4) - (y_{3n}, v_4) + (y_{4n}, v_4) = (f_4(y_{4n}, u_4), v_4),$ 23

$(y_{4n}^0, v_4) = (y_4^0, v_4),$ 24

Where, $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset V_i \subset L^2(\Omega)$ is the projection of y_i^0 w.r.t. the norm $\|.\|_{L^2(\Omega)}$, i.e. $\|y_{in}^0 - y_i^0\|_{L^2(\Omega)} \le \|y_i^0 - v_i\|_{L^2(\Omega)}$, $\forall v_i \in V_n$, $\forall i = 1,2,3,4$.

By utilizing Eq.16 (for i=1,2,3,4) in Eq.17 – Eq.24 resp., and then setting $v_i = v_{il}$, $\forall i = 1,2,3,4$ and $\forall l = 1,2,\dots\dots,n$, one gets that Eq.17 – Eq.24 are equivalent to the following nonlinear system of 1^{st} order ODEs with ICs, which has a unique solution (from the continuity of all the matrices and vectors):

$A_1\grave{C}_1(t) + B_1C_1(t) - DC_2(t) + EC_3(t) + FC_4(t) = b_1\left(\bar{V}_1^T(x)C_1(t)\right),$ 25

$A_1C_1(0) = b_1^0,$ 26

$A_2\grave{C}_2(t) + B_2C_2(t) + KC_1(t) - MC_3(t) - NC_4(t) = b_2\left(\bar{V}_2^T(x)C_2(t)\right),$ 27

$A_2C_2(0) = b_2^0,$ 28

$A_3\grave{C}_3(t) + B_3C_3(t) - PC_1(t) + QC_2(t) + SC_4(t) = b_3\left(\bar{V}_3^T(x)C_3(t)\right),$ 29

$A_3C_3(0) = b_3^0,$ 30

$A_4\grave{C}_4(t) + B_4C_4(t) - XC_1(t) + YC_2(t) - ZC_3(t) = b_4\left(\bar{V}_4^T(x)C_4(t)\right),$ 31

$A_4 C_4(0) = b_4^0$, 32

Where $\forall\, l = 1,2, \dots\dots, n$, $\forall i = 1,2,3,4$.

$A_i = (a_{ilj})_{n\times n}$, $a_{ilj} = (v_{ij}, v_{il})$, $B_i = (b_{ilj})_{n\times n}$, $b_{ilj} = (\nabla v_{ij}, \nabla v_{il}) + (v_{ij}, v_{il})$, $D = (d_{lj})_{n\times n}$, $d_{lj} = (v_{2j}, v_{1l})$, $E = (e_{lj})_{n\times n}$, $e_{lj} = (v_{3j}, v_{1l})$, $F = (f_{lj})_{n\times n}$, $f_{lj} = (v_{4j}, v_{1l})$, $K = (k_{lj})_{n\times n}$, $k_{lj} = (v_{1j}, v_{2l})$, $M = (m_{lj})_{n\times n}$, $m_{lj} = (v_{3j}, v_{2l})$, $N = (n_{lj})_{n\times n}$, $n_{lj} = (v_{4j}, v_{2l})$, $P = (p_{lj})_{n\times n}$, $p_{lj} = (v_{1j}, v_{3l})$, $Q = (q_{lj})_{n\times n}$, $q_{lj} = (v_{2j}, v_{3l})$, $S = (s_{lj})_{n\times n}$, $s_{lj} = (v_{4j}, v_{3l})$, $X = (x_{lj})_{n\times n}$, $x_{lj} = (v_{1j}, v_{4l})$, $Y = (y_{lj})_{n\times n}$, $y_{lj} = (v_{2j}, v_{4l})$, $Z = (z_{lj})_{n\times n}$, $z_{lj} = (v_{3j}, v_{4l})$, $b_i^0 = (b_{il}^0)$, $b_{il}^0 = (y_i^0, v_{il})$, $b_i = (b_{il})_{n\times 1}$ $b_{il} = (f_i(\bar{V}_i^T C_i(t), u_i), v_{il})$, $\bar{V}_i = (v_i)_{n\times 1}$ $\grave{C}_i(t) = (\grave{c}_{il}(t))_{n\times 1}$, $C_i(t) = \left(c_{ij}(t)\right)_{n\times 1}$, $C_i(0) = \left(c_{ij}(0)\right)_{n\times 1}$.

The norm $\|\vec{y}_n^0\|_{L^2(\Omega)}$ is bounded:

Since $y_1^0 = y_1^0(x) \in L^2(\Omega)$, then there is a sequence $\{v_{1n}^0\}$ with $v_{1n}^0 \in V_n$, s.t. $v_{1n}^0 \longrightarrow y_1^0$ strongly (ST) in $L^2(\Omega)$, and from the projection theorem [20], and Eq.26, $\|y_{1n}^0 - y_1^0\|_{L^2(\Omega)} \le \|y_1^0 - v_1\|_{L^2(\Omega)}$, $\forall\, v_1 \in V$, then $\|y_{1n}^0 - y_1^0\|_{L^2(\Omega)} \le \|y_1^0 - v_{1n}^0\|_{L^2(\Omega)}$, $\forall\, v_{1n}^0 \in V_n \subset V$, $\forall\, n$.

Thus, $y_{1n}^0 \longrightarrow y_1^0$ ST in $L^2(\Omega)$ with $\|y_{1n}^0\|_{L^2(\Omega)} \le b_1$. Similarly, once get that $\|y_{2n}^0\|_{L^2(\Omega)} \le b_2$, $\|y_{3n}^0\|_{L^2(\Omega)} \le b_3$ and $\|y_{4n}^0\|_{L^2(\Omega)} \le b_4$, then this implies to $\|\vec{y}_n^0\|_{L^2(\Omega)}$ is bounded.

The norms $\|\vec{y}_n(t)\|_{L^\infty(I, L^2(\Omega))}$ and $\|\vec{y}_n(t)\|_{L^2(Q)}$ are bounded:

Setting $v_i = y_{in}$, $\forall\, i = 1,2,3,4$ in Eq.17 – Eq.24 resp., integration of both sides w.r.t. t from 0 to T, and then adding the obtained four equations together, to get that:

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle\, dt + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt =$
$\int_0^T [\, (f_1(y_{1n}, u_1), y_{1n}) + (f_2(y_{2n}, u_2), y_{2n}) + (f_3(y_{3n}, u_3), y_{3n}) + (f_4(y_{4n}, u_4), y_{4n})]dt$ 33

.

Since $\vec{y}_{nt} \in L^2(I, V^*)$ and $\vec{y}_n \in L^2(I, V)$ then, Lemma 1 can be used for the 1^{st} term in L.H.S. of Eq.33, on the other hand, since the 2^{nd} term is non-negative taking $T = t$, with $t \in I$, finally using Assumptions A-i for the R.H.S. of Eq.33, it yields to:

$\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|^2_{L^2(\Omega)}\, dt \le \|\eta_1\|^2_{L^2(Q)} + \|\eta_2\|^2_{L^2(Q)} + \|\eta_3\|^2_{L^2(Q)} + \|\eta_4\|^2_{L^2(Q)} + \acute{c}_1\|u_1\|^2_{L^2(Q)} + \acute{c}_2\|u_2\|^2_{L^2(Q)} +$
$\acute{c}_3\|u_3\|^2_{L^2(Q)} + \acute{c}_4\|u_4\|^2_{L^2(Q)} + c_9\int_0^t \|\vec{y}_n\|^2_{L^2(\Omega)}\, dt$,

$c_9 = \max(c_5, c_6, c_7, c_8)$.

Where $c_5 = 1 + \acute{c}_1 + 2c_1$, $c_6 = 1 + \acute{c}_2 + 2c_2$, $c_7 = 1 + \acute{c}_3 + 2c_3$, $c_8 = 1 + \acute{c}_4 + 2c_4$.

Since $\|\vec{y}_n(0)\|^2_{L^2(\Omega)} \le b$, $\|\eta_i\|_{L^2(Q)} \le \acute{b}_i$, $\|u_i\|_{L^2(Q)} \le c_i$, $\forall i = 1,2,3,4$, then,

$\|\vec{y}_n(t)\|^2_{L^2(\Omega)} \le c^* + c_9\int_0^t \|\vec{y}_n\|^2_{L^2(\Omega)}\, dt$,

where $c^* = b + b_1' + b_2' + b_3' + b_4' + \acute{c}_1 c_1 + \acute{c}_2 c_2 + \acute{c}_3 c_3 + \acute{c}_4 c_4$.

Using the continuous Bellman Gronwall inequality(CBGI), to obtain:

$\|\vec{y}_n(t)\|^2_{L^2(\Omega)} \le c^* e^{c_9 T} = b^2(c)$, $\forall\, t \in [0, T]$ it implies to $\|\vec{y}_n(t)\|_{L^\infty(I, L^2(\Omega))} \le b(c)$.

Thus, $\|\vec{y}_n(t)\|^2_{L^2(Q)} = \int_0^T \|\vec{y}_n\|^2_{L^2(\Omega)} dt \le b_1^2(c)$, it leads to $\|\vec{y}_n(t)\|_{L^2(Q)} \le b_1(c)$, $b_1^2(c) = Tb^2(c)$.

The norm $\|\vec{y}_n(t)\|^2_{L^2(I,V)}$ is bounded: Again using Lemma 1 for the 1^{st} term in L.H.S. of Eq.33, using the same above results which are obtained from the R.H.S. of Eq.33, then setting $t = T$ and $\|\vec{y}_n(0)\|^2_{L^2(\Omega)}$ is non-negative, easily one gets that:

$\int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt \le \frac{(b + c^* + c_9 b_1(c))}{2} = b_2^2(c) \Rightarrow$
$\|\vec{y}_n\|_{L^2(I,V)} \le b_2(c)$.

The Convergence of the Solution:

Let $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$, s.t. $\forall\, \vec{v} \in \vec{V}$, there is a sequence $\{\vec{v}_n\}$ with $\vec{v}_n \in \vec{V}_n$, $\forall\, n$, and $\vec{v}_n \longrightarrow \vec{v}$ converges strongly in $\vec{V}$ (which gives $\vec{v}_n \longrightarrow \vec{v}$ converges strongly in $L^2(\Omega)$, since for each n, with $\vec{V}_n \subset \vec{V}$, problem Eq.17 – Eq.24 has a unique solution $\vec{y}_n$, hence corresponding to the sequence of subspaces $\{\vec{V}_n\}_{n=1}^{\infty}$, there is a sequence of approximation problems like Eq.17 – Eq.24, now by setting $\vec{v} = \vec{v}_n$, for $n = 1,2, \dots$, one gets that $\forall v_{in} \in V_n$:

$\langle y_{1nt}, v_{1n} \rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n}) = (f_1(y_{1n}, u_1), v_{1n})$, 34

$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n})$, 35

$\langle y_{2nt}, v_{2n} \rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n}) = (f_2(y_{2n}, u_2), v_{2n})$, 36

$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n})$, 37

$\langle y_{3nt}, v_{3n} \rangle + (\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n}) = (f_3(y_{3n}, u_3), v_{3n})$, 38

$(y_{3n}^0, v_{1n}) = (y_3^0, v_{3n}),$ 39

$\langle y_{4nt}, v_{4n}\rangle + (\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n}) = (f_4(y_{4n}, u_4), v_{4n}),$ 40

$(y_{4n}^0, v_{4n}) = (y_4^0, v_{4n}),$ 41

Then Eq.34 – Eq.41 has a sequence of solution $\{\vec{y}_n\}_{n=1}^{\infty}$, but from the previous steps once has $\|\vec{y}_n\|_{L^2(\mathbf{Q})}$ and $\|\vec{y}_n\|_{L^2(\mathrm{I},\mathbf{V})}$ are bounded, then by Alaoglu's theorem, there is a subsequence of $\{\vec{y}_n\}_{n\in\mathbb{N}}$, for simplicity say again $\{\vec{y}_n\}_{n\in\mathbb{N}}$, s.t.: $\vec{y}_n \longrightarrow \vec{y}$ converges weakly in $\boldsymbol{L^2(Q)}$ and $\vec{y}_n \longrightarrow \vec{y}$ converges weakly in $\boldsymbol{L^2(I,V)}$.

At this point, it's necessary to demonstrate that the norm $\|\vec{y}_{kt}\|_{L^2(I,V^*)}$ is bounded, the demonstration of this point will be left here and it will be shown later in Theorem 3, so suppose it is bounded, and since

$$\left(L^2(\mathbb{R},V)\right)^4 \subset \left(L^2(\mathbb{R},\Omega)\right)^4 \cong \left(\left(L^2(\mathbb{R},\Omega)\right)^*\right)^4 \subset \left(L^2(\mathbb{R},\mathrm{V}^*)\right)^4 \qquad 42$$

As a result, the injections of $\left(L^2(\mathbb{R},V)\right)^4$ into $\left(L^2(\mathbb{R},\Omega)\right)^4$, and of $\left(\left(L^2(\mathbb{R},\Omega)\right)^*\right)^4$ into $\left(L^2(\mathbb{R},\mathrm{V}^*)\right)^4$ are continuous, the injection of $\left(L^2(\mathbb{R},V)\right)^4$ into $\left(L^2(Q)\right)^4$ is compact. From Assumptions A, the Cauchy-Schwartz inequality, then the first compactness theorem[19] can be applied here to obtain that there is a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ such that $\vec{y}_n \longrightarrow \vec{y}$ converges strongly in $\boldsymbol{L^2(Q)}$.

Now, consider the weak form Eq.34 – Eq.41 and take any arbitrary $v_i \in V$, so there is a sequence $\{v_{in}\}$, $v_{in} \in V_n$, $\forall n$, s.t. $v_{in} \longrightarrow v_i$ converges strongly in V (then, $v_{in} \longrightarrow v_i$ converges strongly in $L^2(\Omega)$), $\forall i = 1,2,3,4$.

Now, multiplying both sides of Eq.34, Eq.36, Eq.38, and Eq.40 by $\varphi_i \in C^1[0,T], \forall\, i = 1,2,3,4$ resp., with $\varphi_i(T) = 0$, $\varphi_i(0) \neq 0$, then integration both sides w.r.t. t from 0 to T , and then integrating by parts the 1^{st} terms in L.H.S. of each equality, one has:

$$-\int_0^T (y_{1n}, v_{1n})\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n})]\,\varphi_1(t)dt - \int_0^T (y_{2n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{3n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{4n}, v_{1n})\varphi_1(t)dt = \int_0^T (f_1(y_{1n}, u_1), v_{1n})\varphi_1(t)dt + (y_{1n}^0, v_{1n})\varphi_1(0), \quad 43$$

$$-\int_0^T (y_{2n}, v_{2n})\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n})]\,\varphi_2(t)dt + \int_0^T (y_{1n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{3n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{4n}, v_{2n})\varphi_2(t)dt = \int_0^T (f_2(y_{2n}, u_2), v_{2n})\varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0), \quad 44$$

$$-\int_0^T (y_{3n}, v_{3n})\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n})]\,\varphi_3(t)dt - \int_0^T (y_{1n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{2n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{4n}, v_{3n})\varphi_3(t)dt = \int_0^T (f_3(y_{3n}, u_3), v_{3n})\varphi_3(t)dt + (y_{3n}^0, v_{3n})\varphi_3(0), \quad 45$$

$$-\int_0^T (y_{4n}, v_{4n})\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) + (y_{4n}, v_{4n})]\,\varphi_4(t)dt - \int_0^T (y_{1n}, v_{4n})\varphi_4(t)dt + \int_0^T (y_{2n}, v_{4n})\varphi_4(t)dt - \int_0^T (y_{3n}, v_{4n})\varphi_4(t)dt = \int_0^T (f_4(y_{4n}, u_4), v_{4n})\varphi_4(t)dt + (y_{4n}^0, v_{4n})\varphi_4(0), \quad 46$$

Since $\left.\begin{array}{l} v_{in} \longrightarrow v_i \text{ converges strongly in } L^2(\Omega) \\ v_{in} \longrightarrow v_i \text{ converges strongly in } V \end{array}\right\}$, then

$\left\{\begin{array}{l} v_{in}\dot{\varphi}_i \longrightarrow v_i\dot{\varphi}_i \text{ converges strongly in } L^2(Q) \\ v_{in}\varphi_i \longrightarrow v_i\varphi_i \text{ converges strongly in } L^2(I,V) \end{array}\right.$,

And since $y_{in} \longrightarrow y_i$ converges weakly in $L^2(Q)$ and in $L^2(I,V)$, $v_{in} \longrightarrow v_i$ converges weakly in $L^2(\Omega)$ and $y_{in}^0 \longrightarrow y_i^0$ converges weakly in $L^2(\Omega)$ $\forall\, i = 1,2,3,4$, then the following converges hold:

$$-\int_0^T (y_{1n}, v_{1n})\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n})]\,\varphi_1(t)dt - \int_0^T (y_{2n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{3n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{4n}, v_{1n})\varphi_1(t)dt \longrightarrow -\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt, \quad 47$$

$$(y_{1n}^0, v_{1n})\varphi_1(0) \longrightarrow (y_1^0, v_1)\varphi_1(0)\,, \quad 48$$

$$-\int_0^T (y_{2n}, v_{2n})\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n})]\,\varphi_2(t)dt + \int_0^T (y_{1n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{3n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{4n}, v_{2n})\varphi_2(t)dt \longrightarrow -\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt \quad 49$$

$$(y_{2n}^0, v_{2n})\varphi_2(0) \longrightarrow (y_2^0, v_2)\varphi_2(0)\,, \quad 50$$

$$-\int_0^T (y_{3n}, v_{3n})\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n})]\,\varphi_3(t)dt - \int_0^T (y_{1n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{2n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{4n}, v_{3n})\varphi_3(t)dt \longrightarrow -\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt, \quad 51$$

$$(y_{3n}^0, v_{3n})\varphi_3(0) \longrightarrow (y_3^0, v_3)\varphi_3(0)\,, \quad 52$$

$$-\int_0^T (y_{4n}, v_{4n})\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) + (y_{4n}, v_{4n})]\,\varphi_4(t)dt - \int_0^T (y_{1n}, v_{4n})\varphi_4(t)dt + \int_0^T (y_{2n}, v_{4n})\varphi_4(t)dt - \int_0^T (y_{3n}, v_{4n})\varphi_4(t)dt \longrightarrow -\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt \quad 53$$

$(y_{4n}^0, v_{4n})\varphi_4(0) \longrightarrow (y_4^0, v_4)\varphi_4(0)$, 54

On the other hand, since $v_{in} \in V_n$, then $w_{in} = v_{in}\varphi_i \in C[Q]$ for $i = 1,2,3,4$, and $w_{in} \longrightarrow w_i = v_i\varphi_i$ converges strongly in $L^2(Q)$, thus w_{in} is measurable w.r.t. (x,t), and then using Assumptions A-i, and Proposition 1, once has $\int_Q (f_i(y_{in}, u_i)w_{in}\, dxdt$ is continuous w.r.t. (y_{in}, u_i, w_{in}) , since $\vec{y}_n \longrightarrow \vec{y}$ converges strongly in $\boldsymbol{L^2(Q)}$, therefore

$\int_0^T (f_i(y_{in}, u_i), v_{in})\varphi_i(t)dt \longrightarrow$
$\int_0^T (f_i(y_i, u_i), v_i)\varphi_i(t)dt$, $\forall i = 1,2,3,4$.

From the above converges Eq.43 – Eq.54, give:

$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0),$ 55

$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0),$ 56

$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3(y_3, u_3), v_3)\varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0),$ 57

$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4(y_4, u_4), v_4)\varphi_4(t)dt + (y_4^0, v_4)\varphi_4(0),$ 58

Now, we have the following two cases:

Case 1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall\, i = 1,2,3,4$, in Eq.55 – Eq.58, then using IBP for 1^{st} terms in the L.H.S. of each one of the obtained equations, one gets that:

$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt,$ 59

$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2(y_2, u_2), v_2)\varphi_2(t)dt,$ 60

$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3(y_3, u_3), v_3)\varphi_3(t)dt,$ 61

$\int_0^T \langle y_{4t}, v_4\rangle\varphi_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4(y_4, u_4), v_4)\varphi_4(t)dt,$ 62

Then, Eq.59 – Eq.62 give that $\vec{y}$ is a QSVS of the weak form Eq.8 – Eq.15 a.e. on I.

Case 2: Choose $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \neq 0$, $\forall\, i = 1,2,3,4$. Using integration by parts for 1^{st} terms in the L.H.S. of Eq.59, to obtain that:

$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1(y_1, u_1), v_1)\varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0),$ 63

Moreover, by subtracting Eq.55 from Eq.63, one gets that:

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)$, $\varphi_1(0) \neq 0$,
then $(y_1^0, v_1) = (y_1(0), v_1)$, i.e. the IC Eq.9 holds.

In the same above way, one can get that: $(y_i^0, v_i) = (y_i(0), v_i)$, $\forall i = 2,3,4$, i.e. the ICs Eq.11, Eq.13 and Eq.15 hold.

The Strongly Convergence for $\vec{y}_n$:

Substituting $v_i = y_{in}$, $\forall\, i = 1,2,3,4$, in Eq.17, Eq.19, Eq.21 and Eq.23 resp. then, adding the four obtained equalities together and then, integrating both sides of them from 0 to T, to get :

$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n\rangle\, dt + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1(y_{1n}, u_1), v_{1n}) + (f_2(y_{2n}, u_2), v_{2n}) + (f_3(y_{3n}, u_3), v_{3n}) + (f_4(y_{4n}, u_4), v_{4n})]dt$, 64

On the other hand substituting $v_i = y_i$, $\forall\, i = 1,2,3,4$, in Eq.8, Eq.10, Eq.12 and Eq.14 resp. then, adding them together and then, integrating both sides of the obtained equations from 0 to T, to obtain that:

$\int_0^T \langle \vec{y}_t, \vec{y}\rangle\, dt + \int_0^T \|\vec{y}\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1(y_1, u_1), v_1) + (f_2(y_2, u_2), v_2) + (f_3(y_3, u_3), v_3) + (f_4(y_4, u_4), v_4)]dt$,

,

Using Lemma 1 for the 1^{st} terms in the L.H.S. of Eq.64 and Eq.65, they become resp.:

$\frac{1}{2}\|\vec{y}_n(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1(y_{1n}, u_1), v_{1n}) + (f_2(y_{2n}, u_2), v_{2n}) + (f_3(y_{3n}, u_3), v_{3n}) + (f_4(y_{4n}, u_4), v_{4n})]dt,$ 66

$\frac{1}{2}\|\vec{y}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1(y_1,u_1),v_1) + (f_2(y_2,u_2),v_2) + (f_3(y_3,u_3),v_3) + (f_4(y_4,u_4),v_4)]dt,$ 67

Since $\frac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|^2_{L^2(\Omega)} + \int_0^T (\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt = A_1 - B_1 - C_1,$ 68

Where $A_1 = \frac{1}{2}\|\vec{y}_n(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt,$

$B_1 = \frac{1}{2}\big(\vec{y}_n(T), \vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}_n(0) - \vec{y}(0)\big) + \int_0^T \big(\vec{y}_n(t), \vec{y}(t)\big)dt,$

and $C_1 = \frac{1}{2}\big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)\big) + \int_0^T \big(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\big)dt.$

Since $\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0 = \vec{y}(0)$ converges strongly in $\boldsymbol{L^2(\Omega)}$,

$\vec{y}_n(T) \longrightarrow \vec{y}(T)$ converges strongly in $\boldsymbol{L^2(\Omega)}$, 70

Then, from (69) and (70) we have

$$\begin{cases} \big(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)\big) \longrightarrow 0 \\ \big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) \longrightarrow 0 \end{cases}, \qquad 71$$

and

$$\begin{cases} \|\vec{y}_n(0) - \vec{y}(0)\|^2_{L^2(\Omega)} \longrightarrow 0 \\ \|\vec{y}_n(T) - \vec{y}(T)\|^2_{L^2(\Omega)} \longrightarrow 0 \end{cases}. \qquad 72$$

Since $\vec{y}_n \longrightarrow \vec{y}$ converges weakly in $\boldsymbol{L^2(I,V)}$, then,

$\int_0^T \big(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\big)dt \longrightarrow 0\,,$ 73

Again since $\vec{y}_n \longrightarrow \vec{y}$ converges weakly in $\boldsymbol{L^2(Q)}$, From the continuity of integral $\int_0^T (f_i(y_{in},u_i),y_{in})\,dt$ w.r.t. y_i & u_i and $\vec{y}_n \longrightarrow \vec{y}$ converges strongly in $\boldsymbol{L^2(Q)}$, $\forall i = 1,2,3,4$:

$\int_0^T [\,(f_1(y_{1n},u_1),v_{1n}) + (f_2(y_{2n},u_2),v_{2n}) + (f_3(y_{3n},u_3),v_{3n}) + (f_4(y_{4n},u_4),v_{4n})]dt \longrightarrow \int_0^T [\,(f_1(y_1,u_1),v_1) + (f_2(y_2,u_2),v_2) + (f_3(y_3,u_3),v_3) + (f_4(y_4,u_4),v_4)]dt,$ 74

Now, when $n \longrightarrow \infty$ in both sides of Eq.68, one has the following results:

(1) The 1^{st} two terms in the L.H.S. of Eq.68 are tending to zero from Eq.72.

(2) From Eq.64 and Eq.74, one has Equation $A_1 = \int_0^T [\,(f_1(y_{1n},u_1),v_{1n}) + (f_2(y_{2n},u_2),v_{2n}) + (f_3(y_{3n},u_3),v_{3n}) + (f_4(y_{4n},u_4),v_{4n})]dt \longrightarrow \int_0^T [\,(f_1(y_1,u_1),v_1) + (f_2(y_2,u_2),v_2) + (f_3(y_3,u_3),v_3) + (f_4(y_4,u_4),v_4)]dt.$

(3) Equation $B_1 \longrightarrow$ L.H.S. of Eq.70= $\int_0^T [\,(f_1(y_1,u_1),v_1) + (f_2(y_2,u_2),v_2) + (f_3(y_3,u_3),v_3) + (f_4(y_4,u_4),v_4)]dt.$

(4) The 1^{st} two terms in Equation C_1 are tending to zero from Eq.71, and the last term also tends to zero from Eq.73, from these convergences and the results, Eq.68 gives:

$\int_0^T \|\vec{y}_n - \vec{y}\|^2_{\mathcal{H}^1(\Omega)}dt = \int_0^T (\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \xrightarrow{\text{where } n\to\infty} 0$, gives $\vec{y}_n \longrightarrow \vec{y}$ converges strongly in $\boldsymbol{L^2(I,V)}$.

The uniqueness of the Solution:

Let $\vec{y}$ & $\vec{\bar{y}}$ are two QSVS of the WF of the QSVEs Eq.8, Eq.10, Eq.12, and Eq.14, then from Eq.8 one has:

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1,v_1) - (y_2,v_1) + (y_3,v_1) + (y_4,v_1) = (f_1(y_1,u_1),v_1),\ \forall v_1 \in V_1$

$\langle \bar{y}_{1t}, v_1\rangle + (\nabla \bar{y}_1, \nabla v_1) + (\bar{y}_1,v_1) - (\bar{y}_2,v_1) + (\bar{y}_3,v_1) + (\bar{y}_4,v_1) = (f_1(\bar{y}_1,u_1),v_1),\quad \forall v_1 \in V_1$

By subtracting the 2^{nd} equation from the 1^{st} one, then substituting $v_1 = y_1 - \bar{y}_1$, one has that:

$\langle (y_1-\bar{y}_1)_t, y_1-\bar{y}_1\rangle + (y_1-\bar{y}_1, y_1-\bar{y}_1) - (y_2-\bar{y}_2, y_1-\bar{y}_1) + (y_3-\bar{y}_3, y_1-\bar{y}_1) + (y_4-\bar{y}_4, y_1-\bar{y}_1) = (f_1(y_1,u_1) - f_1(\bar{y}_1,u_1), y_1-\bar{y}_1),$ 75

Same ways can use to get:

$\langle (y_2-\bar{y}_2)_t, y_2-\bar{y}_2\rangle + (y_1-\bar{y}_1, y_2-\bar{y}_2) + (y_2-\bar{y}_2, y_2-\bar{y}_2) - (y_3-\bar{y}_3, y_2-\bar{y}_2) - (y_4-\bar{y}_4, y_2-\bar{y}_2) = (f_2(y_2,u_2) - f_2(\bar{y}_2,u_2), y_2-\bar{y}_2),$ 76

$\langle (y_3-\bar{y}_3)_t, y_3-\bar{y}_3\rangle - (y_1-\bar{y}_1, y_3-\bar{y}_3) + (y_2-\bar{y}_2, y_3-\bar{y}_3) + (y_3-\bar{y}_3, y_3-\bar{y}_3) + (y_4-\bar{y}_4, y_3-\bar{y}_3) = (f_3(y_3,u_3) - f_3(\bar{y}_3,u_3), y_3-\bar{y}_3),$ 77

$\langle (y_4-\bar{y}_4)_t, y_4-\bar{y}_4\rangle - (y_1-\bar{y}_1, y_4-\bar{y}_4) + (y_2-\bar{y}_2, y_4-\bar{y}_4) - (y_3-\bar{y}_3, y_4-\bar{y}_4) + (y_4-\bar{y}_4, y_4-\bar{y}_4) = (f_4(y_4,u_4) - f_4(\bar{y}_4,u_4), y_4-\bar{y}_4),$ 78

Adding Eq.75 – Eq.78, using Lemma 1 for the 1^{st} term of the obtained equations, the 2^{nd} term of the L.H.S. of the obtained equation is non-negative, IBS w.r.t. t from 0 to t, then using Assumptions A-ii of the R.H.S. of it, and then by applying the CBGI to obtain that: $\|(\vec{y} - \vec{\bar{y}})(t)\|^2_{L^2(\Omega)} = 0,\ \forall t \in I.$

Again, IBS of the obtained equation w.r.t. t from 0 to T, using the given IC and the above result for the R.H.S. of the equation, one gets that:

$\int_0^T \|\vec{y} - \vec{\bar{y}}\|^2_{\mathcal{H}^1(\Omega)}\, dt \le L\int_0^T \|\vec{y} - \vec{\bar{y}}\|^2_{L^2(\Omega)}\, dt \Rightarrow$

$\int_0^T \|\vec{y} - \vec{\bar{y}}\|^2_{\mathcal{H}^1(\Omega)}\, dt \le 0,$

$\|\vec{y} - \vec{\bar{y}}\|_{L^2(I,V)} = 0$, which implies to $\vec{y} = \vec{\bar{y}}$. ■

Example 1: Let $\Omega=(0,1)\times(0,1)$, $I=[0,1]$, and the QNLPBVP is as given by Eq.1 – Eq.4, with $f_i(x,t,y_i,u_i)=h_i(x,t)+Sin(y_i)+u_i-Sin(\bar{y}_i)-\bar{u}_i$, where $h_i(x,t)$ be a given function, $\forall\, i=1,2,3,4$.
With the boundary and the initial conditions Eq.5 and Eq.6 are given as:
$y_i(x,t)=0$, on Σ and $y_i^0(x)=4(x_1x_2-x_1x_2^2-x_1^2x_2+x_1^2x_2^2)$, on Ω, $\forall\, i=1,2,3,4$.
Since the function f_i satisfies all the assumptions A, for each $i=1,2,3,4$, let $\vec{u}\in \boldsymbol{L^2(Q)}$ be any given QCCCV, then from Thereon 1, the WF Eq.8 – Eq.15, has a unique QSVS $\vec{y}$.

Existence of a Quaternary Classical Continuous Optimal Control:
In this section, the following Theorems and Lemmas will be needed later in the study of the existence of the QCCOCV.
Theorem 2:
a) In addition to Assumptions A, consider that $\vec{y}$ and $\vec{y}+\overrightarrow{\delta y}$ are the QSVs corresponding to the bounded QCCCVs in $\left(L^2(Q)\right)^4$, $\vec{u}$ and $\vec{u}+\overrightarrow{\delta u}$ resp., then
$$\|\delta\vec{y}\|_{\boldsymbol{L}^\infty\left(I,\boldsymbol{L}^2(\boldsymbol{\Omega})\right)}\le M\|\delta\vec{u}\|_{\boldsymbol{L}^2(\boldsymbol{Q})},\ \|\delta\vec{y}\|_{\boldsymbol{L}^2(Q)}\le M\|\delta\vec{u}\|_{\boldsymbol{L}^2(\boldsymbol{Q})}\ ,\ \|\delta\vec{y}\|_{\boldsymbol{L}^2(I,\boldsymbol{V})}\le M\|\delta\vec{u}\|_{\boldsymbol{L}^2(\boldsymbol{Q})}\ ,\ M\in\mathbb{R}^+.$$
b) With Assumptions A, the operator $\vec{u}\longmapsto\vec{y}_{\vec{u}}$ from $\boldsymbol{L^2(Q)}$ in to $L^\infty\left(\boldsymbol{I},\boldsymbol{L^2}(\Omega)\right)$, or into $\boldsymbol{L^2}(I,\boldsymbol{V})$, or into $\boldsymbol{L^2(Q)}$ is continuous.

Proof: Let $\vec{u}\ \&\ \vec{\bar{u}}\in\boldsymbol{L^2(Q)}$, be two given QCCCVs, then by Theorem 1 there exist $\vec{y}\ \&\ \vec{\bar{y}}$ which are their corresponding QSVEs and are satisfied Eq.8 – Eq.15 i.e.:
$\langle\bar{y}_{1t},v_1\rangle+(\nabla\bar{y}_1,\nabla v_1)+(\bar{y}_1,v_1)-(\bar{y}_2,v_1)+(\bar{y}_3,v_1)+(\bar{y}_4,v_1)=(f_1(\bar{y}_1,\bar{u}_1),v_1)$, 79
$(\bar{y}_1(0),v_1)=(y_1^0,v_1)$, 80
$\langle\bar{y}_{2t},v_2\rangle+(\nabla\bar{y}_2,\nabla v_2)+(\bar{y}_1,v_2)+(\bar{y}_2,v_2)-(\bar{y}_3,v_2)-(\bar{y}_4,v_2)=(f_2(\bar{y}_2,\bar{u}_2),v_2)$, 81
$(\bar{y}_2(0),v_2)=(y_2^0,v_2)$, 82
$\langle\bar{y}_{3t},v_3\rangle+(\nabla\bar{y}_3,\nabla v_3)-(\bar{y}_1,v_3)+(\bar{y}_2,v_3)+(\bar{y}_3,v_3)+(\bar{y}_4,v_3)=(f_3(\bar{y}_3,\bar{u}_3),v_3)$, 83
$(\bar{y}_3(0),v_3)=(y_3^0,v_3)$, 84
$\langle\bar{y}_{4t},v_4\rangle+(\nabla\bar{y}_4,\nabla v_4)-(\bar{y}_1,v_4)+(\bar{y}_2,v_4)-(\bar{y}_3,v_4)+(\bar{y}_4,v_4)=(f_4(\bar{y}_4,\bar{u}_4),v_4)$ 85
$(\bar{y}_4(0),v_4)=(y_4^0,v_4)$, 86
By subtracting Eq.8 – Eq.15 from Eq.79 – Eq.86 resp. and setting $y_i=\bar{y}_i-y_i$, $\delta u_i=\bar{u}_i-u_i$,$\forall i=1,2,3,4$, give that:
$\langle\delta y_{1t},v_1\rangle+(\nabla\delta y_1,\nabla v_1)+(\delta y_1,v_1)-(\delta y_2,v_1)+(\delta y_3,v_1)+(\delta y_4,v_1)=(f_1(y_1+\delta y_1,u_1+\delta u_1),v_1)-(f_1(y_1,u_1),v_1)$, 87
$(\delta y_1(0),v_1)=0$, 88
$\langle\delta y_{2t},v_2\rangle+(\nabla\delta y_2,\nabla v_2)+(\delta y_1,v_2)+(\delta y_2,v_2)-(\delta y_3,v_2)-(\delta y_4,v_2)=(f_2(y_2+\delta y_2,u_2+\delta u_2),v_2)-(f_2(y_2,u_2),v_2)$, 89
$(\delta y_2(0),v_2)=0$, 90
$\langle\delta y_{3t},v_3\rangle+(\nabla\delta y_3,\nabla v_3)-(\delta y_1,v_3)+(\delta y_2,v_3)+(\delta y_3,v_3)+(\delta y_4,v_3)=(f_3(y_3+\delta y_3,u_3+\delta u_3),v_3)-(f_3(y_3,u_3),v_3)$, 91
$(\delta y_3(0),v_3)=0$, 92
$\langle\delta y_{4t},v_4\rangle+(\nabla\delta y_4,\nabla v_4)-(\delta y_1,v_4)+(\delta y_2,v_4)-(\delta y_3,v_4)+(\delta y_4,v_4)=(f_4(y_4+\delta y_4,u_4+\delta u_4),v_4)-(f_4(y_4,u_4),v_4)$, 93
$(\delta y_4(0),v_4)=0$, 94
By substituting $v_i=\delta y_i$, in Eq.87 – Eq.94, $\forall i=1,2,3,4$ resp., adding the obtained equations, using Lemma 1 for the 1^{st} term in the L.H.S., give that:
$$\frac{1}{2}\frac{d}{dt}\|\delta\vec{y}\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}+\|\delta\vec{y}\|^2_{\boldsymbol{\mathcal{H}}^1(\boldsymbol{\Omega})}=(f_1(y_1+\delta y_1,u_1+\delta u_1)-f_1(y_1,u_1),\delta y_1)+(f_2(y_2+\delta y_2,u_2+\delta u_2)-f_2(y_2,u_2),\delta y_2)+(f_3(y_3+\delta y_3,u_3+\delta u_3)-f_3(y_3,u_3),\delta y_3)+(f_4(y_4+\delta y_4,u_4+\delta u_4)-f_4(y_4,u_4),\delta y_4),\quad 95$$
Since the 2^{nd} term of Eq.95 is non-negative, integration of both sides w.r.t. t from 0 to t, then taking the absolute value, and then using Assumptions A-ii, it becomes ($\forall t\in[0,T]$):
$$\left\|\overrightarrow{\delta y}(t)\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}\le 2L_1\int_0^t\|\delta y_1\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_1\int_0^T\|\delta u_1\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_1\int_0^t\|\delta y_1\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+2L_2\int_0^t\|\delta y_2\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_2\int_0^T\|\delta u_2\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_2\int_0^t\|\delta y_2\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+2L_3\int_0^t\|\delta y_3\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_3\int_0^T\|\delta u_3\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_3\int_0^t\|\delta y_3\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+2L_4\int_0^t\|\delta y_4\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_4\int_0^T\|\delta u_4\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt+\bar{L}_4\int_0^t\|\delta y_4\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt\ .$$
Thus, $\left\|\overrightarrow{\delta y}(t)\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}\le\tilde{L}_1\left\|\overrightarrow{\delta u}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{Q})}+\tilde{L}_2\int_0^t\left\|\overrightarrow{\delta y}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}dt$, $\tilde{L}_2=\max\{\hat{L}_1,\hat{L}_2,\hat{L}_3,\hat{L}_4\}$,
where $\tilde{L}_1=\max\{\bar{L}_1,\bar{L}_2,\bar{L}_3,\bar{L}_4\}$, $\hat{L}_1=2L_1+\bar{L}_1$, $\hat{L}_2=2L_2+\bar{L}_2$, $\hat{L}_3=2L_3+\bar{L}_3$, $\hat{L}_4=2L_4+\bar{L}_4$.
By using the CBGI one gets:
$$\left\|\overrightarrow{\delta y}(t)\right\|^2_{\boldsymbol{L}^2(\boldsymbol{\Omega})}\le\tilde{L}_1\left\|\overrightarrow{\delta u}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{Q})}e^{\int_0^T\tilde{L}_2dt}=\tilde{L}_1e^{\tilde{L}_2T}\left\|\overrightarrow{\delta u}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{Q})}=M^2\left\|\overrightarrow{\delta u}\right\|^2_{\boldsymbol{L}^2(\boldsymbol{Q})},$$
Hence, $\left\|\overrightarrow{\delta y}(t)\right\|_{\boldsymbol{L}^2(\boldsymbol{\Omega})}\le M\left\|\overrightarrow{\delta u}\right\|_{\boldsymbol{L}^2(\boldsymbol{Q})}$, $t\in[0,T]$
$\Rightarrow\left\|\overrightarrow{\delta y}\right\|_{\boldsymbol{L}^\infty\left(\boldsymbol{I},\boldsymbol{L}^2(\Omega)\right)}\le M\left\|\overrightarrow{\delta u}\right\|_{\boldsymbol{L}^2(\boldsymbol{Q})}$.

Nowsince $\left\|\overrightarrow{\delta y}\right\|^2_{L^2(Q)} \leq \max_{t\in[0,T]}\left\|\overrightarrow{\delta y}(t)\right\|^2_{L^2(\Omega)} \int_0^T dt \leq M^2\left\|\overrightarrow{\delta u}\right\|^2_{L^2(Q)}$.

Then, $\left\|\overrightarrow{\delta y}\right\|_{L^2(Q)} \leq M\left\|\overrightarrow{\delta u}\right\|_{L^2(Q)}$,where $M^2 = TM^2$.

Using the same way used in the above steps for the R.H.S. of Eq.95 with $t = T$, one has that:

$\left\|\overrightarrow{\delta y}(T)\right\|^2_{L^2(\Omega)} + 2\int_0^T\left\|\overrightarrow{\delta y}\right\|^2_{\mathcal{H}^1(\Omega)} dt \leq \tilde{L}_1\left\|\overrightarrow{\delta u}\right\|^2_{L^2(Q)} + \tilde{L}_2 \left\|\overrightarrow{\delta y}\right\|^2_{L^2(Q)}$,

$\left\|\overrightarrow{\delta y}\right\|^2_{L^2(I,V)} \leq M^2\left\|\overrightarrow{\delta u}\right\|^2_{L^2(Q)}$, where $M^2 = (\tilde{L}_1 + \tilde{L}_2M^2)/2$, and

$\left\|\overrightarrow{\delta y}\right\|_{L^2(I,V)} \leq M\left\|\overrightarrow{\delta u}\right\|_{L^2(Q)}$, where M denotes the various constants.

b) From the results of part (a) above, easily one can get that the operator $\vec{u} \longmapsto \vec{y}$ is Lipschitz continuous. ■

Lemma 3:
With Assumption B, the functional $\vec{u} \longmapsto G_0(\vec{u})$ is continuous on $\boldsymbol{L^2(Q)}$.

Proof: By applying Assumption B and Proposition 1, the integral $\int_Q g_{0i}(x,t,y_i,u_i)\,dxdt$ is continuous on $L^2(Q)$, $\forall\, i = 1,2,3,4$, hence $G_0(\vec{u})$ is continuous on $\boldsymbol{L^2(Q)}$. ■

Theorem 3:
Consider the control set is of the form $\vec{W} = W_1 \times W_2 \times W_3 \times W_4$ with $\vec{U}$ is convex and compact, $\vec{W}_A \neq \emptyset$, and f_i,($\forall i = 1,2,3,4$) has the form: $f_i(x,t,y_i,u_i) = f_{i1}(x,t,y_i) + f_{i2}(x,t)u_i$, with $|f_{i1}(x,t,y_i)| \leq \eta_i(x,t) + c_i|y_i|$ and $|f_{i2}(x,t)| \leq k_i$, with $\eta_i \in L^2(Q)$, k_i, $c_i \geq 0$. If g_{0i}, for each $i = 1,2,3,4$ is convex w.r.t. u_i for each fixed (x,t,y_i). Then there exists a QCCOCV for the considered CCOCP.

Proof: From the Assumptions on $U_i \subset \mathbb{R}$, $\forall\, i = 1,2,3,4$ and the Egorov's theorem, one gets that $\vec{W}$ is weakly compact. Since $\vec{W}_A \neq \emptyset$, so there is $\vec{\bar{u}} \in \vec{W}_A$ and there is a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \vec{W}_A$, $\forall k$ s.t.: $\lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\bar{u}}\in\vec{W}_A} G_0(\vec{\bar{u}})$.

Since $\vec{u}_k \in \vec{W}_A$, $\forall k$ but $\vec{W}$ is, there is a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ which converges weakly to some $\vec{u}$ in $\vec{W}$, i.e.: $\vec{u}_k \longrightarrow \vec{u}$ WK in $\boldsymbol{L^2(Q)}$, with $\|\vec{u}_k\|_Q \leq c$, $\forall k$. From Theorem 1 for each QCCCV $\vec{u}_k$, the weak form of the QVSEs has a unique QSVS $\vec{y}_k = \vec{y}_{\vec{u}_k}$ and the norms $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, so by Alaoglu's theorem there is a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ which converges weakly to some $\vec{y}$ w.r.t these norms, i.e.: $\vec{y}_k \longrightarrow \vec{y}$ converges weakly in the spaces $L^\infty\left(I, \boldsymbol{L^2(\Omega)}\right)$, $\boldsymbol{L^2(Q)}$, and in $\boldsymbol{L^2(I,V)}$.

Now, to show that the norm $\|\vec{y}_{kt}\|_{\boldsymbol{L^2(I,V^*)}}$ is bounded:

The weak form Eq.17 – Eq.24 can be rewritten as:

$\langle y_{1kt}, v_1\rangle = -(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) - (y_{3k}, v_1) - (y_{4k}, v_1) + (f_1(y_{1k}, u_{1k}), v_1)$,

$\langle y_{2kt}, v_2\rangle = -(\nabla y_{2k}, \nabla v_2) - (y_{1k}, v_2) - (y_{2k}, v_2) + (y_{3k}, v_2) + (y_{4k}, v_2) + (f_2(y_{2k}, u_{2k}), v_2)$,

$\langle y_{3kt}, v_3\rangle = -(\nabla y_{3k}, \nabla v_3) + (y_{1k}, v_3) - (y_{2k}, v_3) - (y_{3k}, v_3) - (y_{4k}, v_3) + (f_3(y_{3k}, u_{3k}), v_3)$,

$\langle y_{4kt}, v_4\rangle = -(\nabla y_{4k}, \nabla v_4) + (y_{1k}, v_4) - (y_{2k}, v_4) + (y_{3k}, v_4) - (y_{4k}, v_4) + (f_4(y_{4k}, u_{4k}), v_4)$,

By adding the above equalities, integrating both sides from 0 to T, then taking the absolute value, using the Cauchy-Schwartz inequality, and finally using Assumptions A-i, it yields:

$$\begin{aligned}
\left|\int_0^T\langle\vec{y}_{kt}, \vec{v}\rangle dt\right| \leq{}& (\|\nabla y_{1k}\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)})\|v_1\|_{L^2(Q)}\\
&+\big(\|\nabla y_{2k}\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)}\big)\|v_2\|_{L^2(Q)}\\
&+(\|\nabla y_{3k}\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)})\|v_3\|_{L^2(Q)}\\
&+(\|\nabla y_{4k}\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)})\|v_4\|_{L^2(Q)}\\
&+\left[\|\eta_1\|_{L^2(Q)} + c_1\|y_{1k}\|_{L^2(Q)}\right]\|v_1\|_{L^2(Q)} + \left[\|\eta_2\|_{L^2(Q)} + c_2\|y_{2k}\|_{L^2(Q)}\right]\|v_2\|_{L^2(Q)}\\
&+\left[\|\eta_3\|_{L^2(Q)} + c_3\|y_{3k}\|_{L^2(Q)}\right]\|v_3\|_{L^2(Q)} + \left[\|\eta_4\|_{L^2(Q)} + c_4\|y_{4k}\|_{L^2(Q)}\right]\|v_4\|_{L^2(Q)}\\
&+\grave{c}_1\|u_{1k}\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \grave{c}_2\|u_{2k}\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \grave{c}_3\|u_{3k}\|_{L^2(Q)}\|v_3\|_{L^2(Q)}\\
&+\grave{c}_4\|u_{4k}\|_{L^2(Q)}\|v_4\|_{L^2(Q)}
\end{aligned}$$

Since for each $i = 1,2,3,4$, the following are held:

$\|\nabla y_{ik}\|_{L^2(Q)} \leq \|\nabla\vec{y}_k\|_{\boldsymbol{L^2(Q)}} \leq \|\nabla\vec{y}_k\|_{\boldsymbol{L^2(I,V)}}$, $\|v_i\|_{L^2(Q)} \leq \|\vec{v}\|_{\boldsymbol{L^2(Q)}} \leq \|\vec{v}\|_{\boldsymbol{L^2(I,V)}}$, $\|\nabla v_i\|_{L^2(Q)} \leq \|\nabla\vec{v}\|_{\boldsymbol{L^2(Q)}} \leq \|\vec{v}\|_{\boldsymbol{L^2(I,V)}}$, $\|y_{ik}\|_{L^2(Q)} \leq \|\vec{y}_k\|_{\boldsymbol{L^2(Q)}} \leq \|\vec{y}_k\|_{\boldsymbol{L^2(I,V)}} \leq b_i(c)$, $\|u_{ik}\|_{L^2(Q)} \leq \|\vec{u}_k\|_{\boldsymbol{L^2(Q)}} \leq \bar{c}_i$, $\|\eta_i\|_{L^2(Q)} \leq \grave{b}_i$.

Then, the above inequality leads to:

$\left|\int_0^T\langle\vec{y}_{kt}, \vec{v}\rangle dt\right| \leq 4\|\nabla\vec{y}_k\|_{\boldsymbol{L^2(Q)}}\|\nabla\vec{v}\|_{\boldsymbol{L^2(Q)}} + 16\|\vec{y}_k\|_{\boldsymbol{L^2(Q)}}\|\vec{v}\|_{\boldsymbol{L^2(Q)}} + \bar{b}(c)\|\vec{v}\|_{\boldsymbol{L^2(Q)}}$,

where $\bar{b}(c) = \bar{b}_5(c) + \bar{b}_6(c) + \bar{b}_7(c) + \bar{b}_8(c)$, $\bar{b}_5(c) = \grave{b}_1 + c_1b_1(c) + \grave{c}_1\bar{c}_1$, $\bar{b}_6(c) = \grave{b}_2 + c_2b_2(c) + \grave{c}_2\bar{c}_2$, $\bar{b}_7(c) = \grave{b}_3 + c_3b_3(c) + \grave{c}_3\bar{c}_3$

, $\bar{b}_8(c) = \grave{b}_4 + c_4 b_4(c) + \grave{c}_4\bar{c}_4$, setting $\tilde{b}(c) = 20b_2(c) + \bar{b}(c)$, then

$\|\vec{y}_{kt}\|_{L^2(I,V^*)} = sup\frac{\left|\int_0^T\langle\vec{y}_{kt},\vec{v}\rangle dt\right|}{\|\vec{v}\|_{L^2(I,V)}} \le \tilde{b}(c)$, thus $\|\vec{y}_{kt}\|_{L^2(I,V^*)} \le \tilde{b}(c)$, $\forall\, \vec{y}_{kt} \in \vec{V}^*$.

Eq.42 also is held here and gives the injections $L^2(I,V) \subset L^2(Q)$ and $\left(L^2(Q)\right)^* \subset L^2(I,V^*)$ are continuous but the injection $L^2(I,V) \subset L^2(Q)$ is compact. Then by compactness theorem, there is a subsequence of $\{\vec{y}_k\}$ say again $\{\vec{y}_k\}$ s.t. $\vec{y}_k \longrightarrow \vec{y}$ converges strongly in $L^2(Q)$.

Now, since $\forall k$, y_{ik} $(i = 1,2,3,4)$ is the QSVS of Eq.17 – Eq.24 corresponding to the QCCCV u_{ik} ,i.e.:

$\langle y_{1kt}, v_1\rangle + (\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1) - (y_{2k}, v_1) + (y_{3k}, v_1) + (y_{4k}, v_1) = (f_{11}(x,t,y_{1k}) + f_{12}(x,t)u_{1k}, v_1)$, 96

$\langle y_{2kt}, v_2\rangle + (\nabla y_{2k}, \nabla v_2) + (y_{1k}, v_2) + (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{4k}, v_2) = (f_{21}(x,t,y_{2k}) + f_{22}(x,t)u_{2k}, v_2)$, 97

$\langle y_{3kt}, v_3\rangle + (\nabla y_{3k}, \nabla v_3) - (y_{1k}, v_3) + (y_{2k}, v_3) + (y_{3k}, v_3) + (y_{4k}, v_3) = (f_{31}(x,t,y_{3k}) + f_{32}(x,t)u_{3k}, v_3)$, 98

$\langle y_{4kt}, v_4\rangle + (\nabla y_{4k}, \nabla v_4) - (y_{1k}, v_4) + (y_{2k}, v_4) - (y_{3k}, v_4) + (y_{4k}, v_4) = (f_{41}(x,t,y_{4k}) + f_{42}(x,t)u_{4k}, v_4)$, 99

Let $\forall\, i = 1,2,3,4$, $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \ne 0$, rewriting the 1^{st} terms in L.H.S. of Eq.96 – Eq.99, then multiplying both sides of each one by φ_1 , φ_2 , φ_3 and φ_4 resp., integration both sides w.r.t. t from 0 to T, and integration by parts the 1^{st} terms in the L.H.S. of each equality, one gets that:

$-\int_0^T(y_{1k}, v_1)\dot{\varphi}_1(t)dt + \int_0^T[(\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1)]\,\varphi_1(t)dt - \int_0^T(y_{2k}, v_1)\varphi_1(t)dt + \int_0^T(y_{3k}, v_1)\varphi_1(t)dt + \int_0^T(y_{4k}, v_1)\varphi_1(t)dt = \int_0^T(f_{11}(x,t,y_{1k}), v_1)\varphi_1(t)dt + \int_0^T(f_{12}(x,t)u_{1k}, v_1\,\varphi_1(t))dt + (y_{1k}(0), v_1)\varphi_1(0)$, 100

$-\int_0^T(y_{2k}, v_2)\dot{\varphi}_2(t)dt + \int_0^T[(\nabla y_{2k}, \nabla v_2) + (y_{2k}, v_2)]\,\varphi_2(t)dt + \int_0^T(y_{1k}, v_2)\varphi_2(t)dt - \int_0^T(y_{3k}, v_2)\varphi_2(t)dt - \int_0^T(y_{4k}, v_2)\varphi_2(t)dt = \int_0^T(f_{21}(x,t,y_{2k}), v_2)\varphi_2(t)dt + \int_0^T(f_{22}(x,t)u_{2k}, v_2\,\varphi_2(t))dt + (y_{2k}(0), v_2)\varphi_2(0)$, 101

$-\int_0^T(y_{3k}, v_3)\dot{\varphi}_3(t)dt + \int_0^T[(\nabla y_{3k}, \nabla v_3) + (y_{3k}, v_3)]\,\varphi_3(t)dt - \int_0^T(y_{1k}, v_3)\varphi_3(t)dt + \int_0^T(y_{2k}, v_3)\varphi_3(t)dt + \int_0^T(y_{4k}, v_3)\varphi_3(t)dt = \int_0^T(f_{31}(x,t,y_{3k}), v_3)\varphi_3(t)dt + \int_0^T(f_{32}(x,t)u_{3k}, v_3\,\varphi_3(t))dt + (y_{3k}(0), v_3)\varphi_3(0)$, 102

$-\int_0^T(y_{4k}, v_4)\dot{\varphi}_4(t)dt + \int_0^T[(\nabla y_{4k}, \nabla v_4) + (y_{4k}, v_4)]\,\varphi_4(t)dt - \int_0^T(y_{1k}, v_4)\varphi_4(t)dt + \int_0^T(y_{2k}, v_4)\varphi_4(t)dt - \int_0^T(y_{3k}, v_4)\varphi_4(t)dt = \int_0^T(f_{41}(x,t,y_{4k}), v_4)\varphi_4(t)dt + \int_0^T(f_{42}(x,t)u_{4k}, v_4\,\varphi_4(t))dt + (y_{4k}(0), v_4)\varphi_4(0)$, 103

Since $\vec{y}_k \longrightarrow \vec{y}$ WK in $L^2(Q)$ and in $L^2(I,V)$, then Eq.100 – Eq.103, yield to:

$-\int_0^T(y_{1k}, v_1)\dot{\varphi}_1(t)dt + \int_0^T[(\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1)]\,\varphi_1(t)dt - \int_0^T(y_{2k}, v_1)\varphi_1(t)dt + \int_0^T(y_{3k}, v_1)\varphi_1(t)dt + \int_0^T(y_{4k}, v_1)\varphi_1(t)dt \longrightarrow -\int_0^T(y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T[(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T(y_2, v_1)\varphi_1(t)dt + \int_0^T(y_3, v_1)\varphi_1(t)dt + \int_0^T(y_4, v_1)\varphi_1(t)dt$ 104

$-\int_0^T(y_{2k}, v_2)\dot{\varphi}_2(t)dt + \int_0^T[(\nabla y_{2k}, \nabla v_2) + (y_{2k}, v_2)]\,\varphi_2(t)dt + \int_0^T(y_{1k}, v_2)\varphi_2(t)dt - \int_0^T(y_{3k}, v_2)\varphi_2(t)dt - \int_0^T(y_{4k}, v_2)\varphi_2(t)dt \longrightarrow -\int_0^T(y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T[(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T(y_1, v_2)\varphi_2(t)dt - \int_0^T(y_3, v_2)\varphi_2(t)dt - \int_0^T(y_4, v_2)\varphi_2(t)dt$, 105

$-\int_0^T(y_{3k}, v_3)\dot{\varphi}_3(t)dt + \int_0^T[(\nabla y_{3k}, \nabla v_3) + (y_{3k}, v_3)]\,\varphi_3(t)dt - \int_0^T(y_{1k}, v_3)\varphi_3(t)dt + \int_0^T(y_{2k}, v_3)\varphi_3(t)dt + \int_0^T(y_{4k}, v_3)\varphi_3(t)dt \longrightarrow -\int_0^T(y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T[(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T(y_1, v_3)\varphi_3(t)dt + \int_0^T(y_2, v_3)\varphi_3(t)dt + \int_0^T(y_4, v_3)\varphi_3(t)dt$, 106

$-\int_0^T(y_{4k}, v_4)\dot{\varphi}_4(t)dt + \int_0^T[(\nabla y_{4k}, \nabla v_4) + (y_{4k}, v_4)]\,\varphi_4(t)dt - \int_0^T(y_{1k}, v_4)\varphi_4(t)dt + \int_0^T(y_{2k}, v_4)\varphi_4(t)dt - \int_0^T(y_{3k}, v_4)\varphi_4(t)dt \longrightarrow -\int_0^T(y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T[(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T(y_1, v_4)\varphi_4(t)dt + \int_0^T(y_2, v_4)\varphi_4(t)dt - \int_0^T(y_3, v_4)\varphi_4(t)dt$, 107

Since $y_{ik}(0), \forall i = 1,2,3,4$ are bounded in $L^2(\Omega)$ and from the Projection theorem[20], one gets:

$(y_{ik}(0), v_i)\varphi_i(0) \longrightarrow \left(y_i^0, v_i\right)\varphi_i(0), \quad \forall i = 1,2,3,4$ 108

Now, for fixed $(x,t) \in Q$, $w_1(x,t) = v_1(x)\varphi_1(t)$ is fixed, then $w_1 \in L^\infty(I,V) \subset L^2(I,V) \subset L^2(Q)$. Let $v_1 \in C[\bar{\Omega}]$, then $w_1 \in C[\bar{Q}]$is measurable w.r.t. (x,t), now let $\bar{f}_{11}(y_{1k}) = f_{11}(y_{1k})w_1$, then

$\bar{f}_{11}: Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous w.r.t. y_{1k}. So by applying Proposition 1, the $\int_Q f_{11}(y_{1k}) w_1 dxdt$ is continuous w.r.t. y_{1k} but $y_{1k} \longrightarrow y_1$ converges strongly in $L^2(Q)$, hence

$\int_Q f_{11}(y_{1k}) w_1 dxdt \longrightarrow \int_Q f_{11}(y_1) w_1 dxdt$, $\forall w_1 \in C[\bar{Q}]$ 109

Since $u_{1k} \longrightarrow u_1$ WK in $L^2(Q)$ with $|f_{12}| \le k_1$, then

$\int_Q (f_{12}(x,t) u_{1k}, v_1) dxdt \longrightarrow \int_Q (f_{12}(x,t) u_1, v_1) dxdt$ 110

By the same way, one can get the following results (since $y_{ik} \longrightarrow y_i$ converges strongly in $L^2(Q)$, $u_{ik} \longrightarrow u_i$ converges weakly in $L^2(Q)$ and $|f_{i2}| \le k_i$, $\forall i = 2,3,4$):

$\int_Q f_{i1}(y_{ik}) w_i dxdt \longrightarrow \int_Q f_{i1}(y_i) w_i dxdt$, $\forall w_i \in C[\bar{Q}]$ 111

$\int_Q (f_{i2}(x,t) u_{ik}, v_i) dxdt \longrightarrow \int_Q (f_{i2}(x,t) u_i, v_i) dxdt$ 112

Finally, using (Eq.104, Eq.108 (for $i = 1$), Eq.109 & Eq.110) in Eq.100, using (Eq.105, Eq.108 (for $i = 2$), Eq.111 (for $i = 2$) & Eq.112(for $i = 2$)) in Eq.101, also(Eq.106, Eq.108(for $i = 3$), Eq.111&Eq.112(for $i = 3$)) in Eq.102 and finally (Eq.107, Eq.108(for $i = 4$), Eq.111 & Eq.112(for $i = 4$)) in Eq.103**,** to get that:

$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\, \varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_{11}(x,t,y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x,t)u_1, v_1\, \varphi_1(t))dt + (y_1^0, v_1)\varphi_1(0)$,113

$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\, \varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_{21}(x,t,y_2), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x,t)u_2, v_2\, \varphi_2(t))dt + (y_2^0, v_2)\varphi_2(0)$,114

$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\, \varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_{31}(x,t,y_3), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x,t)u_3, v_3\, \varphi_3(t))dt + (y_3^0, v_3)\varphi_3(0)$,115

$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\, \varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_{41}(x,t,y_4), v_4)\varphi_4(t)dt + \int_0^T (f_{42}(x,t)u_4, v_4\, \varphi_4(t))dt + (y_4^0, v_4)\varphi_4(0)$, 116

Thus Eq.113 – Eq.116 are held for each $v_i \in C[\bar{\Omega}]$, but $C[\bar{\Omega}]$ is dense in V, then they are held for every $v_i \in V$, $\forall i = 1,2,3,4$.

Now, we have following two cases:

Case 1: Choose, ($\forall\, i = 1,2,3,4$), $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, then by using integration by parts for the 1^{st} terms in the L.H.S. of each one of Eq.73 – Eq.76, one gets $\forall\, v_i \in V$, $\forall\, i = 1,2,3,4$ that:

$\int_0^T \langle y_{1t}, v_1 \rangle \varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\, \varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_{11}(x,t,y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x,t)u_1, v_1\, \varphi_1(t))dt$, 117

$\int_0^T \langle y_{2t}, v_2 \rangle \varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\, \varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_{21}(x,t,y_2), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x,t)u_2, v_2\, \varphi_2(t))dt$, 118

$\int_0^T \langle y_{3t}, v_3 \rangle \varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\, \varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_{31}(x,t,y_3), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x,t)u_3, v_3\, \varphi_3(t))dt$, 119

$\int_0^T \langle y_{4t}, v_4 \rangle \varphi_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\, \varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_{41}(x,t,y_4), v_4)\varphi_4(t)dt + \int_0^T (f_{42}(x,t)u_4, v_4\, \varphi_4(t))dt$, 120

Give that:

$\langle y_{1t}, v_1 \rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_{11}(x,t,y_1), v_1) + (f_{12}(x,t)u_1, v_1)$,

$\langle y_{2t}, v_2 \rangle + (\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_{21}(x,t,y_2), v_2) + (f_{22}(x,t)u_2, v_2)$,

$\langle y_{3t}, v_3 \rangle + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_{31}(x,t,y_3), v_3) + (f_{32}(x,t)u_3, v_3)$,

$\langle y_{4t}, v_4 \rangle + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_{41}(x,t,y_4), v_4) + (f_{42}(x,t)u_4, v_4)$,

i.e. $\vec{y}$ is satisfied the WF of the QVSEs $\forall \vec{y} \in \vec{V}$, a.e. on I.

Case 2: Choose, ($\forall\, i = 1,2,3,4$), $\varphi_i \in C^1[I]$, s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \ne 0$, then using integration by parts for 1^{st} terms in the L.H.S. of Eq.113 – Eq.116, one has:

$$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\, \varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_{11}(x,t,y_1), v_1)\varphi_1(t)dt + \int_0^T (f_{12}(x,t)u_1, v_1\, \varphi_1(t))dt + (y_1(0), v_1)\varphi_1(0), \quad 121$$

$$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\, \varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_{21}(x,t,y_2), v_2)\varphi_2(t)dt + \int_0^T (f_{22}(x,t)u_2, v_2\, \varphi_2(t))dt + (y_2(0), v_2)\varphi_2(0), \quad 122$$

$$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\, \varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_{31}(x,t,y_3), v_3)\varphi_3(t)dt + \int_0^T (f_{32}(x,t)u_3, v_3\, \varphi_3(t))dt + (y_3(0), v_3)\varphi_3(0), \quad 123$$

$$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\, \varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_{41}(x,t,y_4), v_4)\varphi_4(t)dt + \int_0^T (f_{42}(x,t)u_4, v_4\, \varphi_4(t))dt + (y_4(0), v_4)\varphi_4(0), \quad 124$$

By subtracting Eq.121 – Eq.124 from Eq.116 – Eq.119 resp., one obtains ($\forall i = 1,2,3,4$):
$(y_i^0, v_i)\varphi_i(0) = (y_i(0), v_i)\varphi_i(0)$, $\varphi_i(0) \neq 0$, $\forall\, \varphi_i \in C^1[0,T]$, thus $y_i^0 = y_i(0) = y_i^0(x)$.
Hence $\vec{y}$ is a QSVS of the weak form of the QVSEs (from Case1 and Case2).
Now, to prove $G_0(\vec{u})$ is weakly lower semi continuous (W.L.S.C.) w.r.t. $(\vec{y}, \vec{u})$.
Since $g_{0i}(x,t,y_i,u_i)$ is continuous w.r.t. (y_i, u_i) and $u_i(x,t) \in U_i$ a.e. in Q with U_i is compact. Hence, g_{0i} is satisfied the hypotheses of Lemma 2, (for each $i = 1,2,3,4$), then applying this lemma gives:

$$\int_Q g_{0i}(x,t,y_{ik},u_{ik})\ dxdt \longrightarrow \int_Q g_{0i}(x,t,y_i,u_{ik})\ dxdt, \quad 125$$

Since (for $i = 1,2,3,4$), $g_{0i}(x,t,y_i,u_i)$ is convex and continuous w.r.t. u_i. Hence

$$\int_Q g_{0i}(x,t,y_i,u_i)\, dxdt \leq \lim_{k\to\infty} \inf \int_Q g_{0i}(x,t,y_i,u_{ik})\, dxdt = \lim_{k\to\infty} \inf \int_Q (g_{0i}(x,t,y_i,u_{ik}) - g_{0i}(x,t,y_{ik},u_{ik}))dxdt + \lim_{k\to\infty} \inf \int_Q g_{0i}(x,t,y_{ik},u_{ik})\ dxdt$$

Then, by Eq.125:

$$\int_Q g_{0i}(x,t,y_i,u_i)\ dxdt \leq \lim_{k\to\infty} \inf \int_Q g_{0i}(x,t,y_{ik},u_{ik})\ dxdt\ ,$$

i.e. $G_0(\vec{u})$ is W.L.S.C. w.r.t. $(\vec{y}, \vec{u})$
but $G_0(\vec{u}) \leq \lim_{k\to\infty} \inf G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}\in \vec{W}_A} G_0(\vec{\bar{u}}_k)$, therefore $G_0(\vec{u}) = \min_{\vec{u}\in \vec{W}_A} G_0(\vec{\bar{u}}_k)$.
Then, $\vec{u}$ is a QCCOCV. ■

Example 2: Consider the classical continuous optimal control problem, consisting of the quaternary nonlinear parabolic boundary value problem which is given in Example 1(above), with $f_{i1}(x,t,y_i) = h_i(x,t) - k_i(x,t)\bar{u}_i + Sin y_i - Sin\bar{y}_i$ and $f_{i2}(x,t) = k_i(x,t)u_i$,
where $h_i(x,t)$ and $k_i(x,t)$ are given functions, $\forall\ i = 1,2,3,4$.
The cost function is $G_0(\vec{u}) = \sum_{i=1}^{4} \int_Q [(y_i - \bar{y}_i)^2 + (u_i - \bar{u}_i)^2]dxdt$, with $\vec{U} = [-1,1]^4$.
Since for each $i = 1,2,3,4$, $g_{0i}(x,t,y_i,u_i) = (y_i - \bar{y}_i)^2 + (u_i - \bar{u}_i)^2$, satisfies the assumption B, and $f_{i1}(x,t,y_i)$, $f_{i2}(x,t)$ satisfy all the hypotheses in theorem 3, then there is a quaternary classical continuous optimal control vector.

Conclusions and Discussions:

In this work, the continuous classical optimal control dominated by a quaternary nonlinear parabolic boundary value problem has been studied. Under suitable assumptions, the existence and uniqueness theorem of the quaternary state vector solution for the weak form of the quaternary nonlinear parabolic boundary value problem with a given quaternary classical continuous control vector have been stated and proved via the Galerkin Method, and the first compactness theorem. In addition to the continuity operator between the quaternary state vector solution of the weak form for the quaternary nonlinear parabolic boundary value problem and the corresponding quaternary classical continuous control vector have been proved. The existence of a quaternary classical continuous optimal control vector dominated by the considered quaternary nonlinear parabolic boundary value problem has been stated and proved under suitable assumptions.

The study of the proposed problem is very interesting in the field of applied mathematics since the proposed problem represents a generalization for a heat equation; furthermore, it represents multi objectives problems that have many applications. Also, these results are very important because they give the green light about the ability of solving such problems numerically. This point serves as a future work for the topic.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

Authors' contributions statement:

J. A. A. Al. and W. A. A. Al. contributed to the design and implementation of the research, the proof of the theorems, and the writing of the manuscript.

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السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة

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الخلاصة:

في هذا العمل هدفنا هو دراسة السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة، بوجود شروط مناسبة . تم ذكر نص وبرهان مبرهنة وجود ووحدانية الحل لمتجه الحالة الرباعي المستمر للصيغة الضعيفة لمسائل القيم الحدودية الرباعية الغير خطية المكافئة عندما يكون متجه السيطرة التقليدية المستمرة معلوماً، بواسطة طريقة كاليركن والمبرهنة المرصوصة الاولى . تم برهان عامل الاستمرارية بين متجه الحالة الرباعي المستمر للصيغة الضعيفة لمسالة القيم الحدودية الرباعية المكافئة ومتجه السيطرة التقليدية المستمرة . أيضاً تم برهان مبرهنة وجود متجه رباعي لسيطرة أمثلية تقليدية مستمرة لهذه المسألة بوجود شروط مناسبة.

الكلمات المفتاحية: السيطرة الامثلية التقليدية، دالة الهدف، طريقة كاليركن، استمرارية ليبشتز، مسائل القيم الحدودية المكافئة.