Numerical Investigation, Error Analysis and Application of Joint Quadrature Scheme in Physical Sciences

Saumya Ranjan Jena1* Mitali Madhumita Acharya1 Damayanti Nayak2 Satya Kumar Misra1

1School of Applied Sciences, Department of Mathematics, KIIT Deemed to be University, Bhubaneswar, Odisha, India. 2Humanities and Department of Mathematics, Odisha University of Technology and Research, Bhubaneswar, India.

Received 4/5/2022, Revised 19/1/2023, Accepted 22/1/2023, Published Online First 20/2/2023

Abstract:
In this work, a joint quadrature for numerical solution of the double integral is presented. This method is based on combining two rules of the same precision level to form a higher level of precision. Numerical results of the present method with a lower level of precision are presented and compared with those performed by the existing high-precision Gauss-Legendre five-point rule in two variables, which has the same functional evaluation. The efficiency of the proposed method is justified with numerical examples. From an application point of view, the determination of the center of gravity is a special consideration for the present scheme. Convergence analysis is demonstrated to validate the current method.

Keywords: Center of gravity, Convergence analysis, Density function, Degree of Precision, Joint quadrature, Maclaurin's series.

MSC 2010: 65D30, 65D32

Introduction:
Numerical techniques are the best fit for solving different integral problems. Recently, different types of quadrature rules are used in the field of numerical integration for the benefits of science and technology. The method of mixing quadrature rule is based on constructing a mixed quadrature rule of higher precision by taking the combination of quadrature rules of lower precision. Many authors have developed joint quadrature for numerical evaluation of real definite integrals. The idea for joint quadrature rule is initiated by later on, the extension part for real definite integrals are executed. The electromagnetic field problems, electric circuit problem, hybrid quadrature rule to find the approximate solution of nonlinear Fredholm integral equation with separable kernel and approximate evaluation of real definite integrals are demonstrated. A mixed quadrature rule based on Gaussian quadrature for approximate evaluation of real definite integrals. Let us consider real definite integrals of the form

\[ \int_a^b f(x) \, dx = I \]

Now splitting the range into \( n \) equal parts with step size \( h \)

\[ a = x_0, x_1 = x_0 + h, \ldots, x_n = b \]

\[ \int_a^b f(x) \, dx = I = h \int_0^n f(ph + x_0) \, dp, \]

Where, \( x - x_0 = ph \)

Using forward interpolation method

\[ I = \int_a^b f(x) \, dx = \frac{1}{2} h \left( \frac{n}{2} \right)^2 + \sum \left[ \right] \]

Different Newton’s close type rules such as Trapezoidal rule, Simpson’s \( \frac{1}{3} \) rd rule, Simpson’s \( \frac{2}{3} \) th rule and Weddle’s rule are obtained by taking \( n = 1, 2, 3 \) and 6 respectively. The Gaussian rule is also represented by

\[ \int_a^b f(x) \, dx = I = 0.5(b - a) \int_{-1}^{1} \left[ \frac{(b+c)+(b-a)}{2} \right] dt \]

Where, \([a, b] = [-1, 1] \).

In the double integration the function \( f \) bounded over the square region

\[ \int_a^b \int_{x_0}^{x_1} f(\theta) \, d\theta \, dx \]
\[ R = \{(x, y): a \leq x \leq b, c \leq y \leq d\} \]

Where,
\[
\int_{a}^{b} \int_{c}^{d} f(x, y) dy dx = I = \\
\frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} f(\zeta, \eta) d\zeta d\eta
\]
\[
\eta = 0.5[(d - c)y + (d + c)], \quad \zeta = 0.5[(b - a)x + (b + a)]
\]

Double integrals have various physical and geometric applications, such as area, volume, and moment of inertia about various axes. Certain gravitational effects on an ejected object can be described by the gravitational force on a point particle located at the object's center of mass. The region can be seen as the center of mass vector of that region in the case that the mass density is constant.

When the mass density is constant it cancels out from the numerator and denominator of the center of mass.

Let \( f(x, y) \) is the density function of a distribution of mass \( M \) in a certain domain \( R \)
\[
M = \int_{R} \int \rho(x, y) dxdy
\]

The co-ordinate \((\bar{x}, \bar{y})\) represents the center of gravity,
\[
\bar{x} = \frac{1}{M} \int_{R} \int xf(x, y) dxdy
\]
\[
\bar{y} = \frac{1}{M} \int_{R} \int yf(x, y) dxdy
\]

Encouraged by the excellent performance of these methods, here is a plan for joint quadrature rule for higher precision as well as application in physical sciences for real definite integrals in two variables. Here the novelty of our paper is that a quadrature rule of lower precision seven has constructed taking the joint effort of Lobatto four-point rule and Gauss Legendre - three-point rule each of precision level five which has compared with Gauss Legendre five-point rule with higher precision nine and the former rule produces better approximation than the later even if the lower precision along with same functional evaluations. Several numerical methods like block method, finite element and transformation technique, B-spline collocation and higher degree B-spline and block method are also highly essential to the present topic.

The content of this paper is organized in eight parts. Construction of basic quadrature rules, Gauss Legendre five-point rule and joint quadrature rule are appeared in second, third and fourth part respectively. Fifth part contains error analysis. Numerical results are displayed in sixth part. The application part is focused on in the seventh part and the eighth part contains the conclusions.

**Basic Quadrature Rule**

Let us discuss some basic Gaussian and Lobatto quadrature rules.

**Gauss Legendre Three-point Quadrature Rule**

\[
R_{GL,3}(f) = \int_{-1}^{1} f(x) dx = \left[ \frac{5}{9} f \left( -\sqrt{3/5} \right) + \frac{8}{9} f \left( 0 \right) \right]
\]

**Gauss Legendre Five-point Rule**

\[
R_{GL,5}(f) = \int_{-1}^{1} f(x) dx = \frac{1}{900} \left[ \left( 322 + 13\sqrt{70} \right) f(-\delta) + 5 f(0) + \left( 322 - 13\sqrt{70} \right) f(\delta) \right]
\]

**Lobatto-four-point Quadrature Rule**

\[
\int_{-1}^{1} f(x) dx = \frac{1}{6} \left[ f(-1) + f(1) + 5 \left( f \left( \frac{1}{\sqrt{5}} \right) + f \left( -\frac{1}{\sqrt{5}} \right) \right) \right]
\]

In the double integration the function \( f \) bounded over the square region

**Joint Quadrature Rule**

Let us consider double integration for real definite integral in two variables
\[
\int_{-1}^{1} \int_{-1}^{1} f(x, y) dxdy = I(f)
\]

**Lobatto-four-point Quadrature Rule**

The two variables representation of Lobatto-four-point quadrature rule is obtained by
\[
I(f) = R_{LA}(f) = \left\{ f(-1, -1) + f(-1, 1) + f(-1, -1), f(1, -1) + f(1, 1) + f(-1, 1) + f(1, -1) \right\}
\]

**Gauss Legendre-three-point Quadrature Rule**

The two variables representation of Gauss Legendre-three-point quadrature rule are given by
\[
I(f) = R_{GL,3}(f) = \left\{ f(-1, -1) + f(-1, 1) + f(1, -1) + f(1, 1) \right\}
\]
\[ l(f) \cong R_{GL3}(f) = \frac{1}{81} \left[ 5f \left( \frac{\sqrt{3}}{5} - \frac{\sqrt{3}}{5} \right) + 8f \left( \frac{\sqrt{3}}{5}, 0 \right) + 5f \left( \frac{\sqrt{3}}{5}, 0 \right) + 5f \left( \frac{\sqrt{3}}{5}, 0 \right) \right] + 8 \left\{ 5f \left( 0, -\frac{\sqrt{3}}{5} \right) + 8f (0, 0, 0) + 5f \left( 0, \frac{\sqrt{3}}{5} \right) \right\} + 5 \left\{ 5f \left( \frac{\sqrt{3}}{5}, -\frac{\sqrt{3}}{5} \right) + 8f \left( \frac{\sqrt{3}}{5}, 0 \right) + 5f \left( \frac{\sqrt{3}}{5}, \frac{\sqrt{3}}{5} \right) \right\} \] 

Where, Eq. 13 and Eq. 14 are quadrature rule of precision five. Hence

\[ R_{GL3}(f) + E_{GL3}(f) = l(f) \]  

\[ R_{I4}(f) + E_{I4}(f) = l(f) \]  

Where, \( E_{I4}(f) \) and \( E_{GL3}(f) \) are the errors due to Lobatto-four-point quadrature rule \( R_{I4}(f) \) and Gauss Legendre-three-point quadrature rule \( R_{GL3}(f) \) for approximating \( l(f) \) by Eq. 13 and Eq. 14 respectively.

Rewriting Eq. 12 by Maclaurin’s series

\[ l(f) = \int_{-1}^{1} \left[ \frac{f(x)}{1!} + \frac{x^2 f_2(x)}{2!} + \frac{x^3 f_3(x)}{3!} + \frac{x^4 f_4(x)}{4!} + \frac{x^5 f_5(x)}{5!} + \frac{x^6 f_6(x)}{6!} \right] dx \]

Integrating Eq. 17

\[ l(f) = 4f_{00}(0,0) + \frac{2}{3} f_{20}(0,0) + f_{02}(0,0) + \frac{1}{30} f_{40}(0,0) + \frac{1}{9} f_{22}(0,0) \]

Eq. 13 can also be represented as

\[ R_{I4}(f) = 4f_{00}(0,0) + \frac{2}{3} f_{20}(0,0) + f_{02}(0,0) + \frac{1}{30} f_{40}(0,0) + \frac{1}{9} f_{22}(0,0) \]

\[ + \frac{1}{180} f_{42}(0,0) + f_{24}(0,0) \]

\[ + \frac{52}{75 \times 6!} f_{60}(0,0) + f_{06}(0,0) + \frac{1}{3600} f_{44}(0,0) \]

\[ + \frac{1456}{225 \times 8!} f_{62}(0,0) + f_{26}(0,0) + \frac{84}{125 \times 8!} f_{80}(0,0) + f_{08}(0,0) \]

Equation 16 can be expressed as

\[ R_{GL3}(f) = 4f_{00}(0,0) + \frac{2}{3} f_{20}(0,0) + f_{02}(0,0) + \frac{1}{30} f_{40}(0,0) + \frac{1}{9} f_{22}(0,0) \]

\[ + \frac{1}{180} f_{42}(0,0) + f_{24}(0,0) \]

\[ + \frac{12}{25 \times 6!} f_{60}(0,0) + f_{06}(0,0) + \frac{1}{3600} f_{44}(0,0) \]

\[ + \frac{1456}{225 \times 8!} f_{62}(0,0) + f_{26}(0,0) + \frac{84}{125 \times 8!} f_{80}(0,0) + f_{08}(0,0) \]

Eq. 22 is obtained by associating Eq. 18 and Eq. 21 with Eq. 16

\[ E_{GL3}(f) = l(f) - R_{GL3}(f) \]

\[ = \frac{16}{25 \times 7!} f_{60}(0,0) + f_{06}(0,0) + \frac{64}{75 \times 8!} f_{62}(0,0) + f_{26}(0,0) + \frac{286256}{703125 \times 8!} f_{80}(0,0) + f_{08}(0,0) \]

Eq. 22 is quadrature rule of precision five.

**Gauss-Legendre Five-point Rule**

33-34 developed the compact form of Gauss-Legendre five-point rule.

\[ l(f) = \frac{1}{10000} \left[ \{\alpha f(f(P,-P) + f(P,P)) + 512 f(P,0) + \beta f(P,Q) + f(P,-Q)\} + \{\alpha f(-P,P) + f(-P,-P)\} + \{128 f(0,0) + f(0,-P)\} + \{64 f(0,0) + f(-P,-P)\} \right] \]
512f(−Q, 0) + β[f(−Q, Q) + f(−Q, −Q)]}

23

where

\[ P = \sqrt{\frac{5 + 2\sqrt{10}}{9}}, \quad Q = \sqrt{\frac{5 - 2\sqrt{10}}{9}}, \quad \alpha = (322 - 13\sqrt{70}) \]

and \( \beta = (322 + 13\sqrt{70}) \)

The two variable mathematical representations of Eq. 23 are denoted by Eq. 24

\[ R_{GL}(f) = 4f_{0,0}(0,0) + \frac{2}{3} \left[ f_{0,2}(0,0) + f_{2,0}(0,0) \right] + \frac{1}{30} \left[ f_{0,4}(0,0) + f_{4,0}(0,0) \right] + \frac{1}{180} \left[ f_{0,6}(0,0) + f_{2,4}(0,0) \right] + \frac{4}{71} \left[ f_{6,0}(0,0) + f_{0,6}(0,0) \right] + \frac{4}{91} \left[ f_{6,0}(0,0) + f_{2,4}(0,0) \right] + \frac{1}{151200} \left[ f_{6,4}(0,0) + f_{2,4}(0,0) \right] + \frac{771}{2155 \times 10^7} \left[ f_{0,10}(0,0) + f_{0,10}(0,0) \right] \]

The error \( E_{GL}(f) \) due to Gauss Legendre-five-point quadrature rule is expressed in Eq. 24

\[ E_{GL}(f) = I(f) - R_{GL}(f) = \frac{139}{2155 \times 10^7} \left[ f_{0,10}(0,0) + f_{0,10}(0,0) \right] \]

Construction of Joint (Mixed) Quadrature Rule

Multiplying Eq. 22 by \( \left( \frac{1}{3} \right) \) and adding it to Eq. 20, \( f \) required joint (mixed) quadrature scheme is

\[ I(f) = \frac{1}{7} \left[ 3R_{LA}(f) + 4R_{GL}(f) \right] + \frac{1}{7} \left[ 3E_{LA}(f) + 4E_{GL}(f) \right] \]

Where

\[ I(f) = R_{LAGL}(f) + E_{LAGL}(f) \]

\[ R_{LAGL}(f) = \frac{1}{7} \left[ 3R_{LA}(f) + 4R_{GL}(f) \right] \]

\[ E_{LAGL}(f) = \frac{1}{7} \left[ 3E_{LA}(f) + 4E_{GL}(f) \right] \]

Eqs. 28 and 29 are known as joint rule and error due to joint rule respectively.

**Determination of Error**

The error analysis is governed by Theorem 1

**Theorem 1:**

The absolute error connected with \( R_{LAGL}(f) \) is expressed as

\[ |E_{LAGL}(f)| = |I(f) - R_{LAGL}(f)| \]

\[ = \frac{2048}{525 \times 8!} |f_{6,2}(0,0) + f_{2,6}(0,0)| + \frac{665024}{4921875 \times 8!} |f_{6,0}(0,0) + f_{0,6}(0,0)| \]

**Proof:**

Associating Eqs. 20 and 22 in Eq. 29

**Lemma 1:**

The truncated error bound 36

\[ |E_{LAGL}(f)| \leq \frac{175}{175 \times 7^1} |\eta_2 - \eta_1| \]

Where,

\[ |\eta_1, \eta_2| \in [−1,1] \quad \text{and} \quad M = \max_{−1 \leq x, y \leq 1} |f_{7,0}(x, y) + f_{0,7}(x, y)| \]

**Proof:**

\[ E_{LA}(f) = -\frac{175}{175 \times 7^1} \left[ f_{6,0}(\eta_1) + f_{0,6}(\eta_2) \right] \]

\[ E_{GL}(f) = \frac{175}{175 \times 7^1} \left[ f_{6,0}(\eta_2) + f_{0,6}(\eta_1) \right] \]

\[ E_{LAGL}(f) = \frac{1}{7} \left[ 3E_{LA}(f) + 4E_{GL}(f) \right] \]

\[ |E_{LAGL}(f)| = \frac{64}{175 \times 7^1} \left[ f_{7,0}(\eta_1) + f_{0,7}(\eta_2) \right] \]

\[ \leq \frac{64}{175 \times 7^1} \left[ |\eta_2 - \eta_1| \right] \]

Eq. 30 is known as the error bound at points \( \eta_1 \) and \( \eta_2 \) in the domain \([-1,1]\).

**Corollary 1:**

\[ \frac{128M}{175 \times 7^1} \geq |E_{LAGL}(f)| \]

**Proof:**

From Eq. 30 and \(|\eta_1 - \eta_2| \leq 2\)

\[ \frac{128M}{175 \times 7^1} \geq |E_{LAGL}(f)| \]

**Numerical Verification of the Current Rule**

It is found that joint quadrature rule executes a close approximation to the analytical solution than Gauss Legendre-five-point rules for different integrals with the same twenty-five-point functional evaluations.

**Example 1**

The numerical results along with errors of Example 1 are reported in Table. 1. Analytical result of joint rule and Gauss-Legendre five-point quadrature rule are depicted in Fig 1. The exact results of different integrals are given below referring 10.
\[ I_1 = \int_0^1 \int_0^1 \frac{\sin(\pi x)}{1 + \cos^2(y)} \, dx \, dy = 0.375228493605129 \]
\[ I_2 = \int_0^\pi \int_0^\pi \cos^2(x + y) \, dx \, dy = 0.308425137534042 \]
\[ I_3 = \int_0^\pi \int_0^1 \cos(x) \sin(y^2) \, dx \, dy = 0 \]
\[ I_4 = \int_0^\pi \int_0^1 x^3 \cos(y) \sin(y) \, dx \, dy = 0.125000000000000 \]

Table 1. (Comparison of numerical results of joint quadrature rule \( R_{LAGL3}(f) \) with \( R_{GL5}(f) \) and corresponding error of Example-1)

<table>
<thead>
<tr>
<th>Integral ( I )</th>
<th>Lobatto-4-point rule ( R_{LA}(f) )</th>
<th>GausLegendre-3-point rule ( R_{GL3}(f) )</th>
<th>joint quadrature rule ( R_{LAGL3}(f) )</th>
<th>Error due to joint quadrature rule ( E_{LAGL3}(f) )</th>
<th>GausLegendre-5-point rule ( R_{GL5}(f) )</th>
<th>Error due to GausLegendre-5-point rule ( E_{GL5}(f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_1 )</td>
<td>0.23398168</td>
<td>0.2346846</td>
<td>0.23438339</td>
<td>0.14084510</td>
<td>0.23428366</td>
<td>0.1409448</td>
</tr>
<tr>
<td></td>
<td>8988131</td>
<td>70677885</td>
<td>2810848</td>
<td>0794281</td>
<td>7883072</td>
<td>25722052</td>
</tr>
<tr>
<td>( I_2 )</td>
<td>0.3084251</td>
<td>0.3084251</td>
<td>0.3084251</td>
<td>0.000000</td>
<td>0.3084251</td>
<td>0.000000</td>
</tr>
<tr>
<td></td>
<td>37534042</td>
<td>37534042</td>
<td>37534042</td>
<td>000000000000000</td>
<td>37534042</td>
<td>000000000000000</td>
</tr>
<tr>
<td>( I_3 )</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td></td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>( I_4 )</td>
<td>0.3006602</td>
<td>0.3006602</td>
<td>0.3006602</td>
<td>0.1756602</td>
<td>0.3006602</td>
<td>0.1756602</td>
</tr>
<tr>
<td></td>
<td>34425586</td>
<td>34425586</td>
<td>34425586</td>
<td>34425586</td>
<td>34425586</td>
<td>34425586</td>
</tr>
</tbody>
</table>

Figure 1. Analytical result of joint rule and Gauss-Legendre five-point quadrature rule

In this section, application of joint quadrature rule is implemented to determine the center of gravity \((\bar{x}, \bar{y})\) with different density function \(f(x, y)\). Table 2, reports numerical results for exact and approximate results of center of gravity \(E(\bar{x}, \bar{y})\) and \(A(\bar{x}, \bar{y})\) of Example-2.

\[ M_1 = \int_0^1 \int_0^1 \frac{1}{y + 1} \, dy \, dx, \quad M_2 = \int_0^\ln 4 \int_0^\ln 3 e^{x+y} \, dx \, dy \]
\[ M_3 = \int_0^2 \int_0^2 (y + e^x) \, dy \, dx, \quad M_4 = \int_0^3 \int_0^3 xy \, dy \, dx \]

Table 2. (Numerical results for exact and approximate value of Center of gravity \(E(\bar{x}, \bar{y})\) and \(A(\bar{x}, \bar{y})\) of Example-2)
Discussion:

The joint quadrature method is based on combining two rules of the same precision level to form a higher level of precision. This scheme is based on the mixing of two constituent rule of same precision level seven to develop a higher degree of precision nine. The present scheme may be extended to obtain approximate solution of volume integrals in Mechanics and also surface integral in Physics. The approximate solution of analytic function in complex plane is also obtained by current method. The order of accuracy of various quadrature is obtained as follows:

\[ |E_{L4}(f)| \geq |E_{GL3}(f)| \geq |E_{GL5}(f)| \geq |E_{AGL3}(f)| \]

Convergence analysis is executed to validate the current approach. The present scheme may be extended to obtain approximate solution of moment of inertia, surface integrals and in volume integrals in Mechanics and also in Physics. The approximate solution of analytic function in complex plane is also obtained by current method

Authors’ declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in department of Environmental Engineering, KIIT Deemed to be University.

Authors’ contributions statement:

SR J developed the idea and worked out the numerical part by MATLAB. D N. initiates the manuscript. MM A wrote the manuscript and interpreted the data. SK M wrote another part of the article and revised it. All Authors read the manuscript minutely and approved the final version of their manuscript.

References:


The numerical double integral and the application of the mixed quadrature scheme in physics.

Somia Rangan Jena 1*, Damayanti Naya 2, Satyajit Mondal Asharya 1

1&2 Department of Applied Sciences, KIIT University, Bhubaneswar, Odisha, India.

Abstract: The mixed quadrature method for the numerical solution of double integrals is presented in this research. This method is based on combining two quadrature rules of the same accuracy level to form a higher level of accuracy. The numerical results of the presented method are compared with those obtained by using the high-precision Gauss-Legendre quadrature rule. The proposed method is validated by numerical examples from the point of view of application. The center of gravity is considered as a special feature of the current quadrature scheme. The convergence analysis is used to verify the reliability of the presented method.

Key terms: Center of gravity, Convergence analysis, Density function, Accuracy degree, Mixed quadrature, MacLaurin series.