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Newton-Kantorovich Method for Solving One of the Non-Linear Sturm-Liouville Problems

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Abstract:

Due to its importance in physics and applied mathematics, the non-linear Sturm-Liouville problems witnessed massive attention since 1960. A powerful Mathematical technique called the Newton-Kantorovich method is applied in this work to one of the non-linear Sturm-Liouville problems. To the best of the authors' knowledge, this technique of Newton-Kantorovich has never been applied before to solve the non-linear Sturm-Liouville problems under consideration. Accordingly, the purpose of this work is to show that this important specific kind of non-linear Sturm-Liouville differential equations problems can be solved by applying the well-known Newton-Kantorovich method. Also, to show the efficiency of applying this method to solve these problems, a comparison is made in this paper between the Newton-Kantorovich method and the Adomian decomposition method applied to the same non-linear Sturm-Liouville problems under consideration in this work. As a result of this comparison, the results of the Newton-Kantorovich method agreed with the results obtained by applying Adomian's decomposition method.

Keywords: Adomian's decomposition method, Central finite-difference approximation, Newton-Kantorovich method, Non-linear Sturm-Liouville problems, Non-Linear differential equations.

Introduction:

Many researchers have shown an interest in the non-linear Sturm-Liouville problems along the time since 1960, and it has applications in physics and applied mathematics. To illustrate this interest, some of the recent research work that has been done regarding these problems will be mentioned here as follows: One of the most recent research works about the Sturm-Liouville problems is the work presented in 2022 by Aal-Rkhais, Kamil, and Oweidi[1] in which they discussed solving the second order singular Sturm-Liouville equation. Also, many interesting theories regarding the non-linear eigenvalue problems of the Sturm-Liouville Type are presented by Kurseeva, Moskaleva, and Valovik[2] in 2019 including deriving solvability results, asymptotics of positive and negative eigenvalues, and also applications were given. Moreover, among other recent works that have to be mentioned here is that presented by He and Yang[3] in 2019 and by Al-Khaled and Hazaimeh[4] in 2020, wherein the work of He and Yang, the existence of positive solutions for systems of non-linear Sturm-Liouville differential equations with weight functions was studied, while in the work of Al-Khaled and Hazaimeh a comparative study between a modified version of the variational iteration method and the Sinc-Galerkin method was presented to solve non-linear Sturm-Liouville eigenvalue problem. In this paper, the Newton-Kantorovich method is applied to approximate the solution for one of the non-linear Sturm-Liouville problems that are the problem: $-y''(x) + y^2(x) = \lambda\, y(x);\ y(x) > 0, x \in I = (0,1)$ subject to the boundary conditions $y(0) = y(1) = 0$ where $\lambda > 0$ is an eigenvalue parameter. Newton - Kantorovich method is an important numerical method that is used to solve some non-linear ordinary and partial differential equations. This mathematical technique is very well-known and there is a massive number of research about it. One of the most recent studies in this regard is the work of Regmi, Argyros, George and Argyros[5], published in 2022, in which they introduced a technique that extended the convergence region of previous studies and they achieved other advantages without additional conditions. Also, by using the Newton-Kantorovich method, Boichuk and Chuiko[6] in 2021 proposed a new iterative scheme for the determination of solutions to the weakly non-linear boundary value problem for a system of ordinary differential equations in the critical case. This paper is not to analyze the convergence of the Newton-Kantorovich method but is devoted to showing the ability to solve a specific kind of non-linear Sturm-Liouville problems using the Newton-Kantorovich method. In 2014, Hussan and Abbas[7] applied the Newton-Kantorovich method to solve a diffusion and exothermic equation, they applied the Newton-Kantorovich method to convert the non-linear boundary value problem into a linear boundary value problem, and then they used the finite difference method to solve the linear boundary value problem. There are different methods to handle non-linear differential equations, some of the most recent of them included in the references[8-11]. In this paper, the same procedure included in the work of Hussan and Abbas[7] is followed.

By using the Adomian's decomposition method, among other results Sennar Somali and Guzin Gokmen[12] obtained the series form for the solution of non-linear Sturm-Liouville problems: $-y''(x) + y^p(x) = \lambda\, y(x);\ y(x) > 0, x \in I = (0,1)$ Subject to the boundary conditions $y(0) = y(1) = 0$ where $\lambda > 0$ is an eigenvalue parameter and $p > 1$ is a constant. The results of the Newton-Kantorovich method of this work are compared with those obtained when applying the series form mentioned above using $\lambda = 9.8696$ and $a = 2.240 \times 10^{-8}$. For other recent applications of the Adomian decomposition method see the work of Al-Jizani and Al-Delfi[13].

Newton-Kantorovich Method for Second Order Differential Equations:

Following the same procedure included in the work of Hussan and Abbas[7], the Newton-Kantorovich method is derived here for solving the single non-linear second order differential equation:

$$y''(x) + f(x, y, y') = 0,\ x \in (a, b), \tag{1}$$

subject to the linear homogeneous boundary conditions $y(a) = y(b) = 0$, by applying the Taylor series expansion. Expanding the function $f(x, y, y')$ using a Taylor series expansion up through first-order term around the solution y_k, gives the following expansion:

$$f(x, y_{k+1}, y'_{k+1}) = f(x, y_k, y'_k) + \frac{\partial f(x,y_k,y'_k)}{\partial y}(y_{k+1} - y_k) + \frac{\partial f(x,y_k,y'_k)}{\partial y'}(y'_{k+1} - y'_k) \tag{2}$$

Now, putting Eq.2 into Eq.1 yields:

$$y''_{k+1} + f(x, y_k, y'_k) + \frac{\partial f(x,y_k,y'_k)}{\partial y}(y_{k+1} - y_k) + \frac{\partial f(x,y_k,y'_k)}{\partial y'}(y'_{k+1} - y'_k) = 0$$

$$y''_{k+1} + \frac{\partial f(x,y_k,y'_k)}{\partial y}(y_{k+1} - y_k) + \frac{\partial f(x,y_k,y'_k)}{\partial y'}(y'_{k+1} - y'_k) = -f(x, y_k, y'_k) \tag{3}$$

To simplify, Eq.3, one can substitute $\delta y_k = y_{k+1} - y_k$ and the following equation is obtained:

$$(\delta y_k)'' + \frac{\partial f(x,y_k,y'_k)}{\partial y'}(\delta y_k)' + \frac{\partial f(x,y_k,y'_k)}{\partial y}(\delta y_k) = -y''_k - f(x, y_k, y'_k) \tag{4}$$

Subject to the boundary conditions:

$\delta y_k(a) = 0$

$\delta y_k(b) = 0$

Now, by giving an initial solution y_0, Eq. 4 can be solved using the finite difference method to get the solution δy_0 at specified mesh points. Therefore, the second solution $y_1 = \delta y_0 + y_0$ can be obtained at the specified mesh points. Hence, putting $k = 1$ and substituting $y_1(x_i)$ at specified mesh points in Eq. 4 and solving by the finite difference method gives the solution at specified mesh points. In a similar way, one can get the third solution $y_2 = \delta y_1 + y_1$ at the same mesh points. Continue in this manner until $\|y_{k+1} - y_k\| < \varepsilon$ at the same mesh points, where ε is a known small positive number.

Application of the Newton-Kantorovich Method to One of the Non-linear Sturm-Liouville Problems:

Consider the non-linear Sturm-Liouville problem:

$$-y''(x) + y^2(x) = \lambda\, y(x);\ y(x) > 0, x \in I = (0,1) \tag{5}$$

$y(0) = y(1) = 0$

when $\lambda > 0$ is an eigenvalue parameter.

One can rewrite Eq. 5 as:

$y''(x) - y^2(x) + \lambda\, y(x) = 0,$

by comparing with Eq. 1, one can have:

$f(x, y, y') = -y^2(x) + \lambda\, y(x).$

Then, the partial derivatives of function f with respect to y' and y respectively will be as follow:

$\frac{\partial f(x,y_k,y'_k)}{\partial y'} = 0,$

$\frac{\partial f(x,y_k,y'_k)}{\partial y} = -2y + \lambda.$

Now, applying the Newton-Kantorovich method to the non-linear Sturm-Liouville problem in Eq. 5 yields:

$(\delta y_k)'' - (2y_k - \lambda)(\delta y_k) = -y_k'' + y_k^2 - \lambda\, y_k,$
6
subject to the homogeneous boundary conditions:
$\delta y_k(0) = 0,$
$\delta y_k(1) = 0,$
where y_k is a previous iteration that is considered to be a known function, and $\delta y_k = y_{k+1} - y_k$.
At $k = 0$, then Eq. 6 will be written as below:
$(\delta y_0)'' - (2y_0 - \lambda)(\delta y_0) = -y_0'' + y_0^2 - \lambda\, y_0.$
Let us choose the first iteration $y_0 \equiv 1$, this reduces the above equation to the following equation:
$(\delta y_0)'' - (2 - \lambda)(\delta y_0) = 1 - \lambda.$
And for simplicity, one can write $\delta = \delta y_0$ in the above equation to rewrite it in the simplest form:
$\delta'' - (2 - \lambda)\delta = 1 - \lambda,$
7
$\delta(0) = 0,$
$\delta(1) = 0.$
Now, applying the central finite-difference approximation on Eq. 7 gives the following difference equation:
$\frac{\delta_{i+1} - 2\delta_i + \delta_{i-1}}{h^2} - (2 - \lambda)\delta_i = 1 - \lambda.$
Multiplying both sides by h^2 to get:
$\delta_{i+1} - (2 + 2h^2 - \lambda h^2)\delta_i + \delta_{i-1} = (1 - \lambda)h^2.$
Take $n = 10$ then $i = 1,2,3,\cdots,9$ and $h = \frac{1}{n} = 0.01$.
Consider the value $\lambda = 9.8696$, see Sennar Somali and Guzin Gokmen[12], then:
$\delta_{i+1} - (1.9213)\delta_i + \delta_{i-1} = -0.0887 \;\; ; \;\; \forall i = 1,2,3,\cdots,9.$
8
Thus, by evaluating Eq. 8 at $i = 1,2,3,\cdots,9$ one can get the following matrix form:
$A\delta = B.$
Where:

$$A = \begin{bmatrix} -1.9213 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1.9213 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1.9213 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.9213 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1.9213 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1.9213 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.9213 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.9213 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1.9213 \end{bmatrix},$$

$$\delta^T = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6 \quad \delta_7 \quad \delta_8 \quad \delta_9],$$

$$B = \begin{bmatrix} -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \\ -0.0887 \end{bmatrix}.$$

So,

$$A^{-1} = \begin{bmatrix} -1.7797 & -2.4193 & -2.8685 & -3.0920 & -3.0722 & -2.8105 & -2.3277 & -1.6617 & -0.8649 \\ -2.4193 & -4.6482 & -5.5113 & -5.9407 & -5.9025 & -5.3998 & -4.4722 & -3.1925 & -1.6617 \\ -2.8685 & -5.5113 & -7.7204 & -8.3218 & -8.2684 & -7.5642 & -6.2647 & -4.4722 & -2.3277 \\ -3.0920 & -5.9407 & -8.3218 & -10.0481 & -9.9835 & -9.1332 & -7.5642 & -5.3998 & -2.8105 \\ -3.0722 & -5.9025 & -8.2684 & -9.9835 & -10.9129 & -9.9835 & -8.2684 & -5.9025 & -3.0722 \\ -2.8105 & -5.3998 & -7.5642 & -9.1332 & -9.9835 & -10.0481 & -8.3218 & -5.9407 & -3.0920 \\ -2.3277 & -4.4722 & -6.2647 & -7.5642 & -8.2684 & -8.3218 & -7.7204 & -5.5113 & -2.8685 \\ -1.6617 & -3.1925 & -4.4722 & -5.3998 & -5.9025 & -5.9407 & -5.5113 & -4.6482 & -2.4193 \\ -0.8649 & -1.6617 & -2.3277 & -2.8105 & -3.0722 & -3.0920 & -2.8685 & -2.4193 & -1.7797 \end{bmatrix}.$$

Then:

$$\delta = A^{-1}B = \begin{bmatrix} 1.8535 \\ 3.4725 \\ 4.7294 \\ 5.5255 \\ 5.7980 \\ 5.5255 \\ 4.7294 \\ 3.4725 \\ 1.8535 \end{bmatrix}.$$

Since $y_1 = \delta y_0 + y_0$, hence one can get the solution y_1 by applying $y_1 = \delta + 1$ which is:

$$y_1 = \begin{bmatrix} 2.8535 \\ 4.4725 \\ 5.7294 \\ 6.5255 \\ 6.7980 \\ 6.5255 \\ 5.7294 \\ 4.4725 \\ 2.8535 \end{bmatrix}.$$

A Series Form for the Solution of the Non-linear Sturm-Liouville Problems by Adomian's Decomposition Method:

In 2007, Sennur Somali and Guzin Gokmen[12], among other interesting results, applied the Adomian's decomposition method to obtain the solution to the non-linear Sturm-Liouville problem:

$$-y'' + y^p(x) = \lambda\, y(x);\ y(x) > 0, x \in I = (0,1)$$

subject to the boundary conditions $y(0) = y(1) = 0$ where $p > 0$ is constant and $\lambda > 0$ is an eigenvalue parameter, in a series form as below:

$$y(x{:}\,\lambda) = \frac{a}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\right)x + \frac{a^p x^{p+2}}{(p+1)(p+2)} + \frac{p a^{2p-1} x^{2p+3}}{(p+1)(p+2)(2p+2)(2p+3)} - \frac{\lambda a^p x^{p+4}}{(p+3)(p+4)}\left(\frac{p}{3!} + \frac{1}{(p+1)(p+2)}\right) + \cdots$$

At $p = 2$ the above non-linear Sturm-Liouville problem is reduced to Eq. 5, and according to the series form above, its solution will be:

$$y(x{:}\,\lambda) = \frac{a}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\right)x + \frac{a^2x^4}{12} + \frac{a^3x^7}{252} - \frac{\lambda a^2 x^6}{30}\left(\frac{5}{12}\right) + \cdots$$

Using this series form when $\lambda = 9.8696$ and $a = 2.240 \times 10^{-8}$, one can find the solution $y(x)$ at the following values:

$$x = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \end{bmatrix}.$$

Thus, the solution is:

$$\Rightarrow y(x) = \begin{bmatrix} 2.2033 \times 10^{-9} \\ 4.1910 \times 10^{-9} \\ 5.7684 \times 10^{-9} \\ 6.7812 \times 10^{-9} \\ 7.1301 \times 10^{-9} \\ 6.7812 \times 10^{-9} \\ 5.7684 \times 10^{-9} \\ 4.1910 \times 10^{-9} \\ 2.2033 \times 10^{-9} \end{bmatrix}.$$

Results:

In this section, a comparison has been made between the results of the application of the Newton-Kantorovich method and the results of the application of the Adomian's decomposition method for solving the non-linear Sturm-Liouville problem considered in Eq. 5 and it shows that the solutions strongly agreed in both techniques that have been used in this research as shown in Table. 1 and Fig. 1.

Table 1. A comparison between the two techniques used in this research.

i	x_i	$y_1(x_i)$	$y(x_i)$
1	0.1	2.8535	2.2033×10^{-9}
2	0.2	4.4725	4.1910×10^{-9}
3	0.3	5.7294	5.7684×10^{-9}
4	0.4	6.5255	6.7812×10^{-9}
5	0.5	6.7980	7.1301×10^{-9}
6	0.6	6.5255	6.7812×10^{-9}
7	0.7	5.7294	5.7684×10^{-9}
8	0.8	4.4725	4.1910×10^{-9}
9	0.9	2.8535	2.2033×10^{-9}

y_1: The solution from Newton -Kantorovich method, $y(x)$: the series form solution from Adomian's decomposition method.

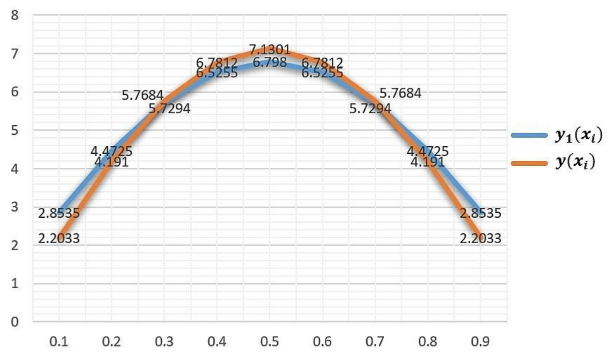

Figure 1. Comparison between $y_1(x_i)$ and $y(x_i)$.

Conclusions:

This paper includes new progress regarding the problem under consideration which is one of the non-linear Sturm-Liouville problems that has importance in physics and applied mathematics. It has been shown here that the Newton-Kantorovich method can be applied to the non-linear Sturm-Liouville problem considered in Eq. 5 and the results of this technique is acceptable since they agree with that one can get when applying another technique called the Adomian's decomposition method for solving the same problem.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah, Iraq.

Authors' contributions statement:

A. made the conception, design, and analysis for this paper while Al. made the drafting, revision, and proofreading and A. made the interpretation, revision, and proofreading.

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طريقة نيوتن - كانتروفيتش لحل إحدى مسائل شتورم - ليوفيل غير الخطية

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الخلاصة:
نظراً لأهميتها في علوم الفيزياء والرياضيات التطبيقية فإن مسائل شتورم – ليوفيل غير الخطية شهدت إهتماماً ملحوظاً منذ عام 1960. واحدة من أهم التقنيات في علم الرياضيات وهي طريقة نيوتن - كانتروفيتش تم تطبيقها في هذا البحث على واحده من مسائل شتورم – ليوفيل غير الخطية. على حد علم المؤلفين لهذا البحث فإن طريقة نيوتن - كانتروفيتش لم يسبق أن تم تطبيقها لحد الآن لحل هذا النوع المهم من مسائل شتورم – ليوفيل غير الخطية التي يتم مناقشة حلها في هذا البحث. وبناءً على ذلك فإن الهدف في هذا البحث هو توضيح أمكانية تطبيق هذه الطريقة المعروفه لحل هذا النوع المهم من مسائل شتورم – ليوفيل غير الخطية. ولتوضيح كفاءة هذه الطريقة في حل هذا النوع من المسائل فقد تم عمل مقارنة في هذا البحث بين تطبيق طريقة نيوتن – كانتروفيتش وتطبيق طريقة تجزئة أدومين لحل المسائل نفسها التي يتناولها هذا البحث وكنتيجة لهذه المقارنة تبين أن النتائج التي حصلنا عليها من كلا الطريقتين متوافقه مع بعضها البعض.

الكلمات المفتاحية: طريقة تجزئة أدومين، تقريب الفروقات المنتهية المركزية ، طريقة نيوتن - كانتروفيتش، مسائل شتورم- ليوفيل غير الخطية، المعادلات التفاضلية غير الخطية.