Dismountable and non-Dismountable Spaces via Proximity Space

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Abstract

Most problems studied by researchers were converted into mathematical formulas in the proximity space because this space is easily employed to solve the different problems of life. Therefore, the current study attempts to find a new property for proximity spaces using the proximity and cluster properties and linking them to obtain new cumulative points, it is called the Bushy Points and the set of all these points is the Bushy set. They form a coherent mathematical foundation for the topological formula that is looking for, which can be called dismountable space. If the space does not have this property, it can be called a non-dismountable space.

Keywords: Attached space, Bushy set, Scant set, Sparse set, Strongly non-dismountable.

Introduction

The essential focus in building the mathematical axis in this work is the proximity space, first blocks were laid by Reazi 1. Then, the focus that emerged from it is the cluster family, which was built on the properties of proximity founded by Lodato 1. Then the events followed and intersected in order to highlight the properties of these spaces and their basic features, among them, for example, Efremovic, Leader, Smirnov, Fenstad, and Njastad. The last focus that relied on is the end points, which are considered an important applied mathematical starting point in the course of engineering and life problems experienced.

Researchers played an essential role in supporting this aspect with many mathematical concepts within the proximity space. Hence, some researchers used the proximity space and linked it with the ideal concept in order to find a new type of end point 2-5. Others tried to employ the proximity space in another direction, which is creating new sets and constructing set theory (σ-Algebra) and then a topological space based on those sets6-9. Other contemporary researchers tried to enrich this space with many interesting concepts10-13.

The current study presents special sets, studies their properties and use them to find concepts to solve and obtain topological properties based on these sets.

Results and Discussion

The following are the basic definitions related to the proximity spaces and clusters and some theorems needed in this work:

**Definition 1:** 1 The ordered pair \((X, \delta)\) represents a proximity space if the following intuitive is satisfied the axioms

P1) If \(A \delta B\), then \(B \delta A\);
P2) \(A \delta (B \cup H)\) if and only if \(A \delta B\) or \(A \delta H\);
P3) \(A \delta B\), then \(A \neq \emptyset\) and \(B \neq \emptyset\);
P4) If $A \cap B$, then $\Delta B$;

P5) If $A \Delta B$, then there exists $E \in \mathcal{P}(X)$ such that $A \Delta B$ and $X \in B \Delta B$; (Efremovich axiom)

P6) $A \Delta B$ and $b \delta H$ for every $b \in B$, then $A \Delta H$ (Lodato axiom).

Definition 2: A subfamily of the proximity space is called a cluster denoted by $\sigma$ if and only if it satisfies the following three conditions:

C1) For all $A, B \in \sigma \Rightarrow \Delta B$;

C2) $(A \cup B) \in \sigma \Rightarrow A \in \sigma$ or $B \in \sigma$;

C3) $A \Delta B$ for each $B \in \sigma \Rightarrow A \in \sigma$.

Proposition 1: Let $(X, \delta)$ be a proximity space. $A, B$ nonempty subsets of $X$, then

1) For any $E \in X$ either $E \in \sigma$ or $E \in \sigma$.

2) If $E \in \sigma$ and $A \in E$, then $B \in \sigma$.

3) $\sigma_{A X} = \{ A \subset X ; \Delta X \}$ is cluster point;

4) If $B \subseteq H$ and $\Delta B \Rightarrow \Delta B$;

Theorem 1: Let $(X, \delta, \chi)$ and $(Y, \delta, \chi)$ be two proximity spaces then a function $f: X \to Y$ is $\delta - \chi$ continuous if and only if $A \Delta B \Rightarrow f(A) \Delta f(B)$ for each $A, B \subseteq X$.

Theorem 2: Let $f$ be a proximity mapping from $(X, \delta, \chi)$ to $(Y, \delta, \chi)$. Then for each cluster $\delta X$ in $X$, there corresponds a cluster $\delta Y$ in $Y$ such that $f(\delta X) = \{ A \subset Y ; \Delta Y \}$ for every $B \in \delta X$.

Definition 3: Let $\sigma$ a cluster in a proximity space $(X, \delta)$, the binary relation $\approx_{\sigma}$ defined on $(X, \delta)$ as follows:

$A \approx_{\sigma} B \Leftrightarrow (A - B) \cup (B - A) \in \sigma$ or $A \approx_{\sigma} B \Leftrightarrow (A \cup B) - (B \cap A) \in \sigma$.

Definition 4: Let $(X, \delta)$ be a proximity space, $\sigma$ is cluster defined on $X$. Then $\sigma$ - Topological Proximity denoted by $\tau_{\sigma}$ is the family nonempty subsets of $X$ satisfies the conditions:

1) $X, \emptyset \in \tau_{\sigma}$;

2) For every subcollection $\mathcal{U}$ of $\tau_{\sigma}$ there exists $\mathcal{V} \in \tau_{\sigma}$ such that $\mathcal{U} \mathcal{V} \approx_{\sigma} \mathcal{V}$;

3) For every $\mathcal{U}, \mathcal{V} \in \tau_{\sigma}$ there exists $\mathcal{W} \in \tau_{\sigma}$ such that $(\mathcal{U} \cap \mathcal{V}) \approx_{\sigma} \mathcal{W}$;

4) $\tau_{\sigma} \cap \sigma = X$.

The pair $(X, \sigma)$ is called proximity space, that is, if $(X, \delta)$ is proximity space, and $\tau$ is any topology defined on $X$ and $\sigma$ is a cluster defined on $(X, \delta)$, then the quadruple $(X, \tau, \delta, \sigma)$ is called proximity cluster topological space. So that, the quadruple pair $(\sigma, \tau, \delta, \sigma)$ is denoted as $\sigma -$ Topological Proximity Space.

Definition 5: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A point $x \in X$ is said to be a follower point of a subset $A$ of topological space $(X, \tau)$, if and only if for every $\mathcal{U} \in \tau (x)$, and every $C \in \sigma$ such that $(\mathcal{U} \cap A) \notin C$, where $\tau (x)$ is the set of all open neighborhood of point $x$. All the follower points of a set $A$ is denoted by $A_{f_{\sigma}}$.

Definition 6: Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. A point $x \in X$ is said to be takeoff point of a subset $A$ of topological space $(X, \tau)$, if and only if there exist $\mathcal{U} \in \tau (x)$, such that $(\mathcal{U} \cap A) \notin C$ for some $C \in \sigma$. All the takeoff points of a set $A$ is denoted by $A_{t_{\sigma}}$.

Example 1: Let $X = \{1,2,3\}$, $\delta$ is discrete proximity $(A \delta B) \Leftrightarrow A \cap B = \emptyset$ (See 1), $\sigma = \{\{1,2,3\}, \{\{1,2\}, \{2,3\}\}, \{\{1\}, \{1,2\}\}, \{\{1,3\}\}$. Let $\tau = \{X, \emptyset, \{1\}, \{1,2\}\}, \{\{1,3\}\}$, $A = \{\{2,3\}, B = \{1\}, H = X$.

1) $\notin A_{f_{\sigma}}$ because $\{1\} \in \tau (1)$, and $\{\{1\} \cup \{2,3\}\}$ for every $C \in \sigma$ since $\{\{1\} \cup \{2,3\}\}$ = $\emptyset$ and $\emptyset \notin C \in \sigma$. Also, $\notin A_{f_{\sigma}}$ because $\{3\} \in \tau (3)$, and $\{2\} \in \sigma$, but $\{\{1\} \cup \{2,3\}\} \notin \{2\}$. Thus $A_{f_{\sigma}} = \{2\}$ because for every $\mathcal{U} \in \tau (2)$, and every $C \in \sigma$ we have $(\mathcal{U} \cap \{2,3\}) \notin \sigma$. Similar of $B$ and $H$, that is $B_{t_{\sigma}} = \emptyset$ and $H_{t_{\sigma}} = \{2\}$. So that $A_{t_{\sigma}} = X$ and $H_{t_{\sigma}} = X$.

Bushy Set and Bushy Spaces

The topological space is known to depend on the neighborhood of the point, and therefore these points
play an important role in finding important topological properties and features within this space. Researchers in this work are trying to introduce points characterized by the fact that all points of space, and all of their neighborhoods, are cluster elements.

**Definition 7:** Let \((X, \tau, \delta, \sigma)\) be a proximity cluster topological space. \(A\) be a nonempty subset of \(X\) is called *bushy set* denoted by \(\mathcal{B}(A)\) if and only if for every \(x \in X\) and every \(\mathcal{U} \in \tau(x)\), \(\mathcal{U} \cap A \delta C\) for every \(C \in \sigma\), and these points are called *bushy points*.

The family of all bushy set on \(X\) denoted by \(\mathcal{B}(X, \delta)\). Moreover, if the \(\delta\) define of \(X\) is discrete proximity \(A \delta B \iff A \cap B \neq \emptyset\) (See 1), then \(\mathcal{B}(X, \delta) \neq \emptyset\) since there exists \(x \in C\) for every \(C \in \sigma\).

**Example 2:** Let \(X = \{1, 2, 3\}\), \(\delta\) be a discrete proximity, \(\sigma = \{(1),\{1,2\},\{1,3\},X\}\), and let \(\tau = \{X, \emptyset,\{1\},\{1,2\}\}\). Let \(A = \{1,3\}\) and \(B = \{3\}\). Then \(A\) is a bushy set because every \(x \in X\) and every \(\mathcal{U} \in \tau(x)\), \(\mathcal{U} \cap A \delta C\) for every \(C \in \sigma\). But \(B\) is not bushy set since there exists \(1 \in X\) and \(\emptyset \in \tau(1)\) such that \(\{1,2\} \cap \{3\} \delta C\) for every \(C \in \sigma\).

**Proposition 2:** Let \((X, \tau, \delta, \sigma)\) be a proximity cluster topological space. Then:

1. Every super set of the bushy set is a bushy set.
2. Finite union of bushy sets is also a bushy set.
3. Every bushy set of \(X\) is member of cluster. This mean \(\mathcal{B}(X, \delta)\) is subset of cluster.

It is to be noted that, according to earlier results, \(X\) is not always a bushy set, but if the space has at least one subset of \(X\) is a bushy set, then \(X\) is a bushy set.

**Proposition 3:** Let \((X, \delta, \sigma, \tau_\sigma)\) be a \(\sigma\)-Topological Proximity space. Then

\[
\mathcal{B}(X, \delta) = \begin{cases} \mathcal{P}(X) / \emptyset & \text{if}\ \delta \text{ is indiscrete} \\ \emptyset & \text{otherwise} \end{cases}
\]

**Proof:** Case one: If \(\delta\) is indiscrete proximity, then \(\tau_\sigma = \{X, \emptyset\}\). and \(\sigma = \{A \subseteq X; A \neq \emptyset\}\). Then only nonempty \(\tau_\sigma = \text{open}\) set is \(X\) for every nonempty subset \(A\) of \(X\), that is \((X \cap A) \delta C\) for every \(C \in \sigma\).

Hence \(\mathcal{B}(X, \delta) = \mathcal{P}(X) / \emptyset\).

Case two: If possible there exists nonempty subset \(A\) of \(X\) is bushy set, then every proper \(\mathcal{U} \in \tau_\sigma\), \((\mathcal{U} \cap A) \delta C\) for every \(C \in \sigma\), by axiom \([C3]\) \(\mathcal{U} \cap A \neq \emptyset\), thus \(\mathcal{U} \in \sigma\) which is a contradiction with \(\tau_\sigma \cap \sigma = \{X\}\). Hence \(\mathcal{B}(X, \delta) = \emptyset\).

**Remark 1:** Let \((X, \tau, \delta, \sigma)\) be a proximity cluster topological space. If \(A\) is a bushy set, then \(A\) is a dense set.

**Proof:** Let \(A\) be a bushy set. Then for every \(x \in X\) and every \(\mathcal{U} \in \tau(x)\), \(\mathcal{U} \cap A \delta C\) for every \(C \in \sigma\), that is, \(\mathcal{U} \cap A \neq \emptyset\) for every \(\mathcal{U} \in \tau(x)\). Hence \(A\) is a dense set.

**Remark 2:** Let \((X, \tau_1, \delta, \sigma)\) and \((X, \tau_2, \delta, \sigma)\) be two proximity cluster topological spaces. If \(\tau_1 \subseteq \tau_2\) and \(A\) be \(\tau_2 - \text{bushy set}\), then \(A\) is \(\tau_1 - \text{bushy set}\).

**Proof:** Let \(A\) is \(\tau_2 - \text{bushy set}\). Then for every \(x \in X\) and every \(\mathcal{U} \in \tau_2(x)\), \(\mathcal{U} \cap A \delta C\) for every \(C \in \sigma\). Since \(\tau_1 \subseteq \tau_2\), \(\mathcal{V} \cap A \delta C\) for every \(\mathcal{V} \in \tau_1\). Hence \(A\) is \(\tau_1 - \text{bushy set}\).

**Definition 8:** The proximity cluster topological space is called a *bushy space* if and only if every bushy subset of \(X\) is an open.

If we take the indiscrete proximity \((A \delta B\) if and only if \(A \neq \emptyset\) and \(B \neq \emptyset\)) and \(\tau\) any topology define on \(X\) and cluster \(\sigma\) defined on \((X, \delta)\), then \(X\) is bushy space. As evidenced by Definition 8, the complement of every bushy set in *bushy space* is a closed set.

**Proposition 4:** If \((X, \tau, \delta, \sigma)\) is a bushy space has at least one subset of \(X\) is a bushy set, then every member of cluster \(\sigma\) is a bushy relative subspace.

**Proof:** Assuming that \(X\) is a bushy space, \(Y \subseteq X\), and \(Y \in \sigma\). Let \(A \subseteq Y\) be a bushy set in \(Y\). Then for every \(\mathcal{V} \in \tau_Y\), \((\mathcal{V} \cap A) \delta C\) for every \(C \in \sigma_Y\), that is, \(\mathcal{A}_f = Y\). It is sufficient to show that \(A\) is \(\tau_Y\)-open such that \(A = \mathcal{U} \cap Y\) for some \(\mathcal{U} \in \tau\).

Let \(\mathcal{U} = A \cup (X - Y)\) \(\Rightarrow\) \(\mathcal{U}_f = \mathcal{A}_f = (A \cup (X - Y))_f = A_f \cup (X - Y)_f\). According to the space has at least one bushy set we have that \(X = X_f\), \((Y \cup (X - Y))_f = Y_f \cup (X - Y)_f\).
(A_f)_{f_a} \cup (X-Y)_{f_a} \subseteq A_{f_a} \cup (X-Y)_{f_a} = U_{f_a},

hence U is bushy set in X, and from X is Bushy space we have that U is open set in X.

So, U \cap Y = (A \cup (X-Y)) \cap Y = A \cap Y = A that is, A is \tau_Y -open in Y i.e., Y is Bushy relative subspace.

The converse don’t always be true, because by Example 2 if we take A = \{1,3\}, then, \tau_A = \{A, \emptyset, \{1\}\} , \{1\} \subset A is \tau_A -bushy set and \tau_A -open , A = \{1,3\} is also \tau_A -bushy and \tau_A -open , hence \tau_A is Bushy space. \{1,3\} \subset X is \tau -bushy set but not is \tau -open set, hence X don’t Bushy space.

**Proposition 5:** If every member of cluster is open, then X is a Bushy space.

**Proof:** Let us suppose that A is bushy set in X. Then for every x \in X and every U \in \tau (x) , U \cap A\delta C for every C \in \sigma , and so U \cap A \in \sigma by Proposition 1 part 2, A \in \sigma . Since every member of cluster is open it follows that A is open set and so X is Bushy space.

**Remark 3:** Let (X, \tau, \delta, \sigma) has F. I. P. (finite intersection property) for every nonempty open sets and every member of cluster is open, A is an open set if and only if A is a bushy set.

**Proof:** Let us suppose that A is an open set by F. I. P. we have that U \cap A \neq \emptyset for every U \in \tau (x).

Since every C \in \sigma is open, (U \cap A) \cap C \neq \emptyset by Proposition 1 part 5 (U \cap A)\sigma C for every C \in \sigma this is true for every x \in X, hence A is bushy set. The Converse is the same as the one given in Proposition 5.

**Proposition 6:** The intersection of any finite family of bushy sets in Bushy space is bushy set.

**Proof:** Let \{A_i\}_{i=1}^n be family of bushy set. Since X is Bushy space then A_i is open set for every i \in \Delta . We prove that by Induction Law. First, to prove the result is true when i = 2. Let A_1 and A_2 are bushy sets. Then for every x \in X and every U \in \tau (x) , U \cap A_1 \delta C for every C \in \sigma . Since X is Bushy space, A_1 \in \tau , that is U \cap A_1 is nonempty open set. Since A_2 is bushy set then for every x \in X and every U \in \tau (x) , U \cap (A_1 \cap A_2)\delta C for every C \in \sigma , hence A_1 \cap A_2 is bushy set. Second, suppose the result is true for i = n - 1. Finally, to prove the result is true for i = n. Let A_n is bushy set. Then for every nonempty U \in \tau , U \cap A_n \delta C for every C \in \sigma . But \{A_i\}_{i=1}^n is nonempty open set this implies U \cap ((\bigcap_{i=1}^{n-1} A_i) \cap A_n)\delta C for every C \in \sigma . Hence \{A_i\}_{i=1}^n is a bushy set.

**Corollary 1:** The intersection of finite family of dense sets in Bushy space is a dense set.

**Proof:** By Remark 1 every bushy set is dense set. Then by Proposition 6 the intersection of finite family of dense sets in Bushy space is a dense set.

**Proposition 7:** If A is bushy set in Bushy space X, then A builds a chain of open and dense subsets of X.

**Proof:** This is an immediate consequence of Proposition 2 part 1, Remark 1 and Definition 8.

**Proposition 8:** If X is Bushy space and \{A_i \subseteq X, i \in N\} is any finite closed cover of X, then int(A_k) \neq \emptyset for some k \in N.

**Proof:** Suppose X = \bigcup_{i=1}^n (A_i). By De-Morgan’s Law, \emptyset = \bigcap_{i=1}^n (X - A_i). Since X is a Bushy space and X - A_i is an open set for every i, clear that by Corollary 1 not every X - A_i is dense set, hence there exist at least one X - A_k is not dense set. Thus there exists nonempty open set U such that U \cap (X - A_k) = \emptyset that is U \subseteq A_k for at least one k, hence int(A_k) \neq \emptyset.

**Proposition 9:** Let f be an open and \delta -continuous proximity mapping from (X, \tau, \delta_x, \sigma) into (Y, \rho, \delta_y, \sigma). If A is bushy set in X, then f(A) is bushy set in Y.

**Proof:** Suppose A is bushy set in X. Then every x \in X and every U \in \tau (x) , \tau (\cap A)\delta C for every C \in \sigma , since f is \delta -continuous this implies f(\bigcap A)\delta f(C) for every f(x) \in Y and every f(U) is neighborhood of f(x) , \tau (\cap A) \subseteq (f(U) \cap f(A))\delta f(C) for every f(C) \in f(\sigma). Since f is open and \delta -continuous , f(U) \in \rho and f(\sigma) is cluster in Y . Thus f(A) is bushy set in Y.

**Proposition 10:** Let f be an injective open \delta -continuous mapping from (X, \tau, \delta_x, \sigma) into (Y, \rho, \delta_y, \sigma). If Y is Bushy space, then X is also Bushy space.

**Proof:** Let A \subseteq X is bushy set in X. By Proposition 9, f(A) is also bushy set in Y. Since Y
is *Bushy space*. \( f(A) \) is \( \rho \)–open. Further, \( f \) is injective continuous this implies \( f^{-1}(f(A)) = A \) is \( \tau \)–open, hence \( X \) is *Bushy space*.

**Definition 9:** The proximity cluster topological space is called *Attached space* if and only if for every nonempty open subset of \( X \) is bushy subset.

**Example 3:** Let \( X = \{1,2,3\}, \delta \) is a discrete proximity, \( \sigma = \{\{3\}, \{1,3\}, \{2,3\}, X\} \). Let \( \tau = \{X, \emptyset, \{3\}, \{2,3\}\} \). Then \( \{2,3\}, \{3\} \) are open and bushy sets, hence \( X \) is attached space.

Note that, if \( \tau \) is a discrete topology, then \( X \) is not attached space since every singleton set is an open but not bushy set. Also can be note that, attached space cannot be separated by two nonempty open subsets because if possible \( X \) can be separated by two nonempty open subsets \( A \) and \( B \) such that \( A \cap B = \emptyset \), \( A \cup B = X \), that is, \( A = X - B \) and \( B = X - A \). Since \( A \) is bushy set, \( A \) is dense set hence \( cl(A) = X \) and so \( B = int(B) = X - cl(X - B) = X - cl(A) = X - X = \emptyset \), which is an contradiction.

In addition, it concludes from the above the following proposition:

**Proposition 11:** Let \((X, \tau, \delta, \sigma)\) be a proximity cluster topological space, the following statements are equivalent:

1- Every nonempty open subset is a bushy set.

2- The cluster \( \sigma \) defines on \((X, \delta)\) containing every nonempty open subset.

**Proof:**

1 \( \Rightarrow \) 2 Let us assume that every nonempty open subset is a bushy set. Let us consider \( G \) is bushy open set. Then every \( x \in X \) and every \( U \in \tau(x) \), \( U \cap G \delta C \) for every \( C \in \sigma \) hence \( G \in \sigma \).

2 \( \Rightarrow \) 1 If possible \( G \in \tau \) is not a bushy set. Then there exists \( U \in \tau(x) \), \( U \cap G \delta C \) for some \( C \in \sigma \) hence \( U \cap G \in \sigma \) this is a contradiction with hypothesis. Thus \( G \) is bushy set.

**Corollary 2:** If the cluster \( \sigma \) containing every nonempty open subset, then the space is attached space.

**Proof:** This is an immediate consequence of Proposition 11 and Definition 9.

Thus we can see that if the space satisfies Remark 3, then the space is *Bushy space* and the Attached space together.

**Definition 10:** Let \((X, \tau, \delta, \sigma)\) be a proximity cluster topological space, a subset \( A \) of \( X \) is called *scant set* if and only if \((A_f)_t = \emptyset \).

By Example 2, the sets \( \{2,3\}, \{2\}, \{3\} \) and \( \emptyset \) are scant set. Evident, if \( \emptyset_t = \emptyset \), then \( \emptyset \) is scant set and if \( A \) is bushy set then \( X - A \) is scant for every \( X - A \in \sigma \). Because \((X - A)_f = \emptyset \) and \[((X - A)_f)_t = X - (X - (X - A)_f)_t = X - (X - \emptyset)_f = X - X_t = X - X = \emptyset \).

It is also clear that the takeoff set in the proximity cluster topology space is an open set. Therefore, this set becomes dense when the space carries an “attached” property.

**Definition 11:** \((X, \tau, \delta, \sigma)\) is non-dismountable space if and only if \( X \) cannot contains two disjoint bushy subsets. Otherwise \( X \) is dismountable space.

It is easy to see that \((X, \delta, \sigma, \tau_\sigma)\) is a non-dismountable space when \( \delta \) is not indiscrete proximity because it doesn’t have any bushy subset in this space. Also, every dismountable space is a resolvable space. Because if \( X \) is a dismountable space, there exist two disjoint bushy subsets, that is, there exist two disjoint dense subsets, hence \( X \) is a resolvable space.

There is a relationship between dense, attached, non-dismountable, and irresolvable space, which is embodied in the following proposition.

**Proposition 12:** Let \((X, \delta, \tau, \sigma)\) is attached space and submaximal. \( X \) is a non-dismountable space if and only if \( X \) is irresolvable space.

**Proof:** If suppose that \( X \) is resolvable. Then there exist two disjoint dense sets \( A, B \) such that \( A \cap B = \emptyset, A \cup B = X \). Since every dense set is open and \( X \) is attached space, \( A \) and \( B \) are bushy sets, that is, \( X \) is dismountable. But this is a contradiction with hypothesis. Conversely, If suppose that \( X \) is dismountable. Then there exist two disjoint bushy sets \( A, B \) such that \( A \cap B = \emptyset, A \cup B = X \). Since every bushy set is dense set, \( A \) and \( B \) are dense sets,
hence $X$ is resolvable, this is a contradiction. Thus $X$ is a non-dismountable space.

**Proposition 13:** Let $\tau_1$ and $\tau_2$ be two topological defined on $X$, such that $\tau_1 \subseteq \tau_2$. If $(X, \tau_1, \delta, \sigma)$ is non-dismountable space, then $(X, \tau_2, \delta, \sigma)$ is non-dismountable space.

**Proof:** Let us suppose that $\tau_2$ is dismountable space. Then there exist two disjoint $\tau_2$–bushy sets $A, B$ such that $A \cap B = \emptyset, A \cup B = X$. By Remark 2 every $\tau_2$–bushy set is $\tau_1$–bushy set, that is $\tau_1$ is dismountable space this contradiction, hence $(X, \delta, \tau_2, \sigma)$ is non-dismountable space.

The above proposition can be generalized to any topology that is finer than the non-dismountable topology defined on the same cluster.

**Definition 12:** The proximity cluster topological space $(X, \tau, \delta, \sigma)$ is called,

1. “Hereditarily non-dismountable if and only if every subspace is non-dismountable.

2. “Strongly non-dismountable if and only if each open subspace is non-dismountable.

For example, if we take a special type proximity, the discrete proximity, then the space $(X, \tau, \delta, \sigma)$:

- Is non-dismountable space.
- The subspace of $X$ is also non-dismountable space.
- Is hereditarily non-dismountable space.
- Is strongly non-dismountable space.

**Proposition 14:** Let $(X, \tau, \delta, \sigma)$ be a proximity cluster topological space. Then every bushy space is non-dismountable space.

**Proof:** If possible that $X$ is a dismountable space. Then there exist two disjoint bushy sets $A, B$ such that $A \cap B = \emptyset, A \cup B = X$, that is $X = A \cap B = X = A$. Since $A$ is bushy set, $A$ is dense set, so that $\text{cl}(A) = X$ but $B = \text{int}(B) = X - \text{cl}(X - B) = X - \text{cl}(A) = X - X = \emptyset$. This is a contradiction with hypothesis. Thus $X$ is non-dismountable space.

**Proposition 15** Let $(X, \tau, \delta, \sigma)$ be a non-dismountable space and for every subset $A$ of $X$ satisfies $A \subseteq A_{\tau \sigma}$. The following statement is equivalent:

1. Every open subspace of $X$ is non-dismountable space.
2. Every takeoff set of bushy set is also bushy set.
3. Every complement of bushy subset of $X$ is scant set.
4. Every closed subset of $X$ is the union of open set and scant set.

**Proof:** $1 \Rightarrow 2$ Suppose every open is non-dismountable. Let $A$ is bushy subset of $X$, that means $A_{\tau \sigma} = X$. If possible $(A_{\tau \sigma})_{\tau \sigma} \neq X$, and $G = X - (A_{\tau \sigma})_{\tau \sigma} \neq \emptyset$. Since $(A_{\tau \sigma})_{\tau \sigma}$ is closed set, $G$ is open set, and so $G \cap A_{\tau \sigma} \subseteq (G \cap A)_{\tau \sigma}$ hence $G = G \cap (G \cap A)_{\tau \sigma}$, that is, $(A \cap G)$ is bushy in $G$. By hypothesis $A_{\tau \sigma} \subseteq (A_{\tau \sigma})_{\tau \sigma}$ thus $G = X - (A_{\tau \sigma})_{\tau \sigma}$, hence $G = X - (A_{\tau \sigma})_{\tau \sigma} = (X - A_{\tau \sigma})_{\tau \sigma}$. Since $G$ is open set, $G = G \cap (X - A)_{\tau \sigma} \subseteq (G \cap X - A)_{\tau \sigma}$ this implies to $G = G \cap (G - A)_{\tau \sigma}$ this means $(G - A)$ is bushy set in $G$. Further, $G = (G \cap A) \cup (G - A)$ and $(G \cap A) \cap (G - A) = \emptyset$, so $(G \cap A), (G - A)$ are bushy set in $G$ hence $G$ is dismountable subspace, which is a contradiction with hypothesis.

$2 \Rightarrow 3$ Suppose $X - A$ is bushy set. By part 2, $(X - A)_{\tau \sigma}$ is dismountable by De-Morgan’s law we have $\emptyset = X - (X - A)_{\tau \sigma} = (A_{\tau \sigma} - A)$, hence $A$ is scant set.

$3 \Rightarrow 4$ Let $\text{int}(A) \in \tau$. Since $\text{int}(A) \subseteq A_{\tau \sigma}$, $A_{\tau \sigma} = \emptyset$ or $A_{\tau \sigma} \neq \emptyset$. In case of $A_{\tau \sigma} = \emptyset$ this implies $\emptyset \subseteq A_{\tau \sigma} = X - (X - A)_{\tau \sigma} \Rightarrow (X - A)_{\tau \sigma} = X$, that is, $(X - A)$ is bushy set. By part 3, $A$ is scant set, hence $A = \emptyset \cup A$ where $\emptyset \in \tau$ and $A$ is scant set. In case of $A_{\tau \sigma} \neq \emptyset$. $(A - A_{\tau \sigma})_{\tau \sigma} = (A \cap X - A_{\tau \sigma})_{\tau \sigma} \subseteq (A_{\tau \sigma} \cap X - A_{\tau \sigma})_{\tau \sigma} \cap (X - A_{\tau \sigma})_{\tau \sigma} = (A_{\tau \sigma} \cap X - A_{\tau \sigma})_{\tau \sigma} = \emptyset$ is also bushy set by De-Morgan’s law we have $(X - A - A_{\tau \sigma})_{\tau \sigma} = X$, this means $(X - A - A_{\tau \sigma})_{\tau \sigma}$ is bushy set $\Rightarrow (A - A_{\tau \sigma})_{\tau \sigma}$ is scant set, and so $A = A_{\tau \sigma} \cup (A - A_{\tau \sigma})_{\tau \sigma}$ is open and $(A - A_{\tau \sigma})_{\tau \sigma}$ is scant set.
4 ⇒ 1 If possible $G$ is nonempty dismountable, there exist $A, B$ subsets of $G$ such that $G = A \cup B$, $A \cap B = \emptyset$ and $G \subseteq A_f \sigma$, $G \subseteq B_f \sigma$. Since $A_f \sigma$ is closed set, $A_f \sigma = U \cup C$ where $U$ is open set and $C$ is scan set, so that $U \neq \emptyset$ otherwise $A_f \sigma = C$, this mean $A_f \sigma$ is scan set $\Rightarrow (A_f \sigma)_{\sigma} = \emptyset$ since $A \subseteq A_f \sigma \Rightarrow (A_f \sigma)_{\sigma} = (A_f \sigma)_{\sigma}$ but $G \subseteq A_f \sigma$ this implies $G \subseteq \{\} \subseteq \emptyset \Rightarrow (A_f \sigma)_{\sigma} = \emptyset$ is contradiction, hence $G \neq \emptyset$. Clear that \( int(U) = U \) and \( int(U) \subseteq int(A) \). Since $A$ is bushy set in $G$, $int(A) \cap B \neq \emptyset \Rightarrow A \cap B \neq \emptyset$ this is a contradiction. Thus $G$ is non-dismountable.

**Conclusion**

One problem that has drawn the attention of researchers is the possibility of finding resolvable and non-resolvable spaces, as the concept of density was used as a criterion in defining these spaces. But the question arises, is it possible to find another criterion parallel to the concept of density, through which it is possible to determine the resolvability of spaces.

**Authors’ Declaration**

- Conflicts of Interest: None.

- Ethical Clearance: The project was approved by the local ethical committee in University of Babylon.

**Authors’ Contribution Statement**

A. and J. conceived of the presented idea. A. developed the theory and performed the computations. A. and J. verified the analytical methods. J. encouraged A. to investigate this notion and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

**References**


