On Semigroup Ideals and Right n-Derivation in 3-Prime Near-Rings

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Abstract:
The current paper studied the concept of right n-derivation satisfying certified conditions on semigroup ideals of near-rings and some related properties. Interesting results have been reached, the most prominent of which are the following: Let \( \mathcal{M} \) be a 3-prime left near-ring and \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \) are nonzero semigroup ideals of \( \mathcal{M} \), if \( d \) is a right n-derivation of \( \mathcal{M} \) satisfies on of the following conditions,

\[
\begin{align*}
(i) & \quad d \left( u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n \right) = 0 \quad \forall \quad u_1 \in \mathcal{A}_1, u_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n \in \mathcal{A}_n; \\
(ii) & \quad d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n) \right) = 0 \\
& \quad \forall u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n; \\
(iii) & \quad d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n) \right) = (u_j, v_j) \quad \forall \quad u_1, v_1 \in \mathcal{A}_1 \\
& \quad , u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n; \\
(iv) & \quad \text{If } d + d \text{ is an } n \text{-additive mapping from } \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n \text{ to } \mathcal{M}; \\
(v) & \quad d \left( u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n \right) \in Z(\mathcal{M}) \quad \forall \quad u_1 \in \mathcal{A}_1, u_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n \in \mathcal{A}_n; \\
(vi) & \quad d \left( (u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n) \right) \in Z(\mathcal{M}) \quad \forall \quad u_1, v_1 \in \mathcal{A}_1 \\
& \quad , u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n;
\end{align*}
\]

Then \( \mathcal{M} \) is a commutative ring.

Keywords: Generalized right derivations, Prime near-ring, Right derivations, Right n–derivations, Semigroup ideals.

Introduction:
A left near-ring is a nonempty set \( \mathcal{M} \) with two binary operations \((+), (\cdot)\) which satisfies (i) \((\mathcal{M}, +)\) is a group that is not necessarily abelian, (ii) \((\mathcal{M}, \cdot)\) is a semi group, (iii) \(a \cdot (b + c) = a \cdot b + a \cdot c\) for each \(a, b, c \in \mathcal{M}\) (recall that when \( \mathcal{M} \) satisfies the right distributive law, \((a + b) \cdot c = a \cdot c + b \cdot c\) for each \(a, b, c \in \mathcal{M}\), then \( \mathcal{M} \) will be called right near-ring ), usually \( \mathcal{M} \) will be 3-prime, if for \(x, y \in \mathcal{M}; x \mathcal{M} y = \{0\} \) implies \(x = 0\) or \(y = 0\). A left near-ring \( \mathcal{M} \) is called zero-symmetric if \(0x = 0\) for all \(x \in \mathcal{M}\) (left distributivity yields \(x0 = 0\) ). \(Z(\mathcal{M})\) will refer to the multiplicative center of \( \mathcal{M} \). Let \(0 \neq \mathcal{A} \subseteq \mathcal{M}\), then \(\mathcal{A}\) is said to be a semigroup ideal of \( \mathcal{M} \) if \( \mathcal{A} \mathcal{M} \subseteq \mathcal{M} \) and \( \mathcal{M} \mathcal{A} \subseteq \mathcal{M} \). For each \(m, n \in \mathcal{M}\), then \((m, n) = m + n - m \cdot n\), \([m, n] = mn - nm \) and \(m \circ n = mn + nm\) will be denoted to the additive commutator, Lie product, and Jordan product, respectively. For more about near-ring, make reference to Pilz'.

Certain mappings, involving some algebraic identities, defined on rings\(^3\) or near-rings\(^4\) and sometimes on an appropriate subset of them, and the effect of these mappings on the algebraic structure of the near-rings, how the near-rings can be converted into rings or commutative rings, was a study project that has attracted the interest of many researchers over the past three decades.

Different types of mappings, such as derivations, generalized derivations, left derivations, homoderivations and multipliers on near-rings or rings have been studied and some related properties have been discussed, see\(^7\)\(^10\). Also, the derivation concepts generalization has been studied by various means according to different authors such as \(n\)-derivations, \((\sigma, \tau)\)-n-derivation, right \(n\)-derivation and generalized right \(n\)-derivation, on near-ring and obtained new interest results for
researchers in this field\textsuperscript{11-14}. Majed and Farhan\textsuperscript{15} are the first to define the concepts of right derivation and right n-derivation on the near-ring.

Let $d$ be an additive mapping from $\mathcal{M}$ into itself, $d$ is said to be a right derivation of $\mathcal{M}$ if $d(mn) = d(m)n + d(n)m$ for each $m, n \in \mathcal{M}$. Let $d: \mathcal{M} \times \mathcal{M} \times \ldots \times \mathcal{M} \to \mathcal{M}$ be $n$-additive mapping (i.e. additive in each argument), $d$ is said to be right $n$-derivation of $\mathcal{M}$ if the following equations hold for each $m_1, m_1', m_2, m_2', \ldots, m_n, n\in\mathcal{M}$:

\[
\begin{align*}
  d(m_1m_1', m_2, \ldots, m_n) &= d(m_1, m_2, \ldots, m_n)m_1' \\
  d(m_1, m_2m_2', \ldots, m_n) &= d(m_1, m_2, \ldots, m_n)m_2' \\
  \vdots & \quad \vdots \\
  d(m_1, m_2, \ldots, m_nn_n') &= d(m_1, m_2, \ldots, m_n)m_n' + d(m_1, m_2, \ldots, m_n') m_n. 
\end{align*}
\]

In this line of inspection, this work will give new essential results in this field and generalize some known results presented.

Note that from now, $\mathcal{M}$ will be 3-prime left near-ring, the abbreviation $\mathcal{CR}$ will refer to the commutative ring while $\mathcal{RD}$ and $\mathcal{RN}D$ are a brief to the right derivation and right $n$-derivation respectively.

**Preliminaries**

**Lemma 1:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Proof:** For any $a, b \in \mathcal{M}$, we have $a + b = b + a$, and $\mathcal{M}$ is commutative.

**Corollary 2:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Lemma 2:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Proof:** For any $a, b \in \mathcal{M}$, we have $a + b = b + a$, and $\mathcal{M}$ is commutative.

**Corollary 3:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Lemma 3:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

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**Lemma 4:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Proof:** For any $a, b \in \mathcal{M}$, we have $a + b = b + a$, and $\mathcal{M}$ is commutative.

**Corollary 5:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Lemma 5:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Proof:** For any $a, b \in \mathcal{M}$, we have $a + b = b + a$, and $\mathcal{M}$ is commutative.

**Corollary 6:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Lemma 6:** Let $\mathcal{M}$ be a 3-prime semigroup. Then $\mathcal{M}$ is abelian.

**Proof:** For any $a, b \in \mathcal{M}$, we have $a + b = b + a$, and $\mathcal{M}$ is commutative.
Lemma 0.3: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{B} \subseteq \mathcal{A} \).

Lemma 2: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{A} = \mathcal{B} \).

Lemma 3: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic.

Lemma 4: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{A} \) and \( \mathcal{B} \) have the same cardinality.

Corollary 6: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{A} \) and \( \mathcal{B} \) are isomorphic.

Corollary 7: For any \( \mathcal{A} \) and \( \mathcal{B} \), if \( d(\mathcal{A}, \mathcal{B}) = 0 \), then \( \mathcal{A} \) and \( \mathcal{B} \) have the same cardinality.

Main Results:

Theorem 1: Let \( \mathcal{A} \) be a nonzero semigroup ideal of \( M \) and \( d \) be a nonzero simigroup ideals of \( M \). If \( d(\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n) = 0 \), then either \( \mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \) are isomorphic.

Proof: By assumption

\[
d(d(\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n)) = 0
\]

for any \( \mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \) such that \( d(\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n) = 0 \). Hence \( \mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n \) are isomorphic.

Corollary 10: Let \( \mathcal{A} \) be a nonzero semigroup ideal of \( M \) and \( \mathcal{B} \) be a nonzero semigroup ideal of \( M \). If \( d(\mathcal{A}, \mathcal{B}) = 0 \), then either \( \mathcal{A} \) or \( \mathcal{B} \) is isomorphic.

Proof: By assumption

\[
d(d(\mathcal{A}, \mathcal{B})) = 0
\]

for any \( \mathcal{A} \) and \( \mathcal{B} \) such that \( d(\mathcal{A}, \mathcal{B}) = 0 \). Hence \( \mathcal{A} \) or \( \mathcal{B} \) is isomorphic.
Proof: By assumption
\[ d((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)) = 0 \]
for any
\[ u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n, \]
then \( \mathcal{M} \) is a C.R.

Thus, for any
\[ u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n, \]

Put \((p_j, q_j)\) instead of \(s\) in last equation and use Eq.3 to get \((u_j, v_j)(p_j, q_j) = 0\) for any \(u_j, v_j, p_j, q_j \in \mathcal{A}_j\). It follows \(0 = (u_j, v_j)(mp_j, mq_j) = (u_j, v_j)m(p_j, q_j)\) for any \(u_j, v_j, p_j, q_j \in \mathcal{A}_j, m \in \mathcal{M}\), three primeness of \(\mathcal{M}\) implies \((u_j, v_j) = 0\) for any \(u_j, v_j \in \mathcal{A}_j\). Hence \((\mathcal{M}, +)\) abelian by Lemma 4, consequently \(\mathcal{M}\) is a C.R. by Lemma 9.

Corollary 15: Let \(d\) be a nonzero \(\mathcal{R}, n.D\) of \(\mathcal{M}\) and \(\mathcal{A}\) is a nonzero semigroup ideal of \(\mathcal{M}\), if
\[ d(u_1, u_2, \ldots, (u_j, v_j), \ldots, u_n, v_n) = (u_j, v_j) \]
for any \(u_1, u_2, \ldots, u_n, v_n \in \mathcal{A}\), then \(\mathcal{M}\) is a C.R.

Corollary 16: Let \(d\) be a nonzero \(\mathcal{R}.D\) of \(\mathcal{M}\) and \(\mathcal{A}\) is a nonzero semigroup ideal of \(\mathcal{M}\), if \(d(u_1, u_2, \ldots, v_j, v_j) = (u_j, v_j)\) for any \(u_1, u_2, \ldots, u_n, v_n \in \mathcal{A}\), then \(\mathcal{M}\) is a C.R.

Theorem 3: Let \(d\) be a nonzero \(\mathcal{R}.n.D\) of \(\mathcal{M}\) and \(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\) are nonzero semigroup ideals of \(\mathcal{M}\), if
\[ d((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)) = 0 \]
for any \(u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n, \)

Proven: By hypothesis,
\[ d((u_1, v_1), (u_2, v_2), (u_j, v_j), \ldots, (u_n, v_n)) = (u_j, v_j) \]
for any
\[ u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n. \]

Thus, for any
\[ u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n, \]

Put \((p_j, q_j)\) instead of \(s\) in last equation and use hypothesis to get \((u_j, v_j)(p_j, q_j) = 0\) for any
Corollary 19: Let \( d \) be a nonzero \( R \cdot n \cdot D \) of \( M \), if
\[
d \left( (s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n) \right) = (s_j, t_j)
\]
for any \( s_1, t_1, s_2, t_2, \ldots, s_j, t_j, \ldots, s_n, t_n \in M \), then \( M \in C.R. \).

Theorem 5: Let \( d \) be a \( R \cdot n \cdot D \) of \( M \) and 
\( A_1, A_2, \ldots, A_n \) are nonzero semigroup ideals of \( M \), if \( d + d \) is an \( n \cdot n \) -additive mapping from \( A_1 \times A_2 \times \ldots \times A_n \) to \( M \), then \( M \in C.R. \).

Proof: From hypothesis: For any \( u_1 \in A_1, u_2 \in A_2, \ldots, u_n \in A_n \)
\[
(\text{d} + \text{d})(u_1, u_2, \ldots, u_n) = (\text{d} + \text{d})(u_1, u_2, \ldots, u_n) = d(u_1, u_2, \ldots, u_n)
\]
As well,
\[
(\text{d} + \text{d})(u_1, u_2, \ldots, u_n) = d(u_1, u_2, \ldots, u_n)
\]
Comparing the last two expressions to conclude
\[
d(u_1, u_2, \ldots, u_n) = 0,
\]
the required result obtained by Theorem 1.

Corollary 20: Let \( d \) be a nonzero \( R \cdot n \cdot D \) of \( M \) and 
\( A \) is a nonzero semigroup ideals of \( M \), if \( d + d \) is an \( n \cdot n \) -additive mapping from \( A \times A \times \ldots \times A \) to \( M \), then \( M \in C.R. \).

Corollary 21: Let \( d \) be a nonzero \( R \cdot n \cdot D \) of \( M \), if 
\( d + d \) is an \( n \cdot n \) -additive mapping on \( M \), then \( M \in C.R. \).

Corollary 22: Let \( d \) be a nonzero \( R \cdot D \) of \( M \) and 
\( A \) is a nonzero semigroup ideals of \( M \), if \( d + d \) is an additive on \( A \) and then \( M \in C.R. \).

Corollary 23: Let \( d \) be a nonzero \( R \cdot D \) of \( M \), if \( d + d \) is an additive on \( M \) then \( M \in C.R. \).

Theorem 6: Let \( d_1 \) and \( d_2 \) are two nonzero \( R \cdot n \cdot D's \) of \( M \) (\( M \) is two torsion free) and 
\( A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n \) are nonzero semigroup ideals of \( M \), if 
\[
d_1(A_1, A_2, \ldots, A_n) d_2(B_1, B_2, \ldots, B_n) \subseteq Z(M).
\]
then \( M \in C.R. \).

Proof: By assumption: for any \( a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n, b_1 \in B_1, b_2 \in B_2, \ldots, b_1 \in B_1, b_2 \in B_2, \ldots, b_n \in B_n \),
\[
d_1(a_1, a_2, \ldots, a_n) d_2(b_1, b_2, \ldots, b_n) \subseteq Z(M),
\]
Therefore
\[
d_1(a_1, a_2, \ldots, a_n) d_2(b_1, b_2, \ldots, b_n) = d_1(a_1, a_2, \ldots, a_n) d_2(b_1, b_2, \ldots, b_n)
\]
Corollary 25: Let $d_1$ and $d_2$ are two nonzero \( R \cdot n \cdot D \)'s of \( M \) (\( M \) is a two torsion free) and \( A \) is a nonzero semigroup ideals of \( M \), if $d_1(\mathcal{A}, \mathcal{A}, ..., \mathcal{A}) d_2(\mathcal{A}, \mathcal{A}, ..., \mathcal{A}) \subseteq Z(M)$, then $M$ is C. R. 

Corollary 26: Let $d_1$ and $d_2$ are two nonzero \( R \cdot D \)'s of \( M \) (\( M \) is a two torsion free), \( A \) and \( B \) are nonzero semigroup ideals of \( M \), if $d_1(\mathcal{A}) d_2(\mathcal{B}) \subseteq Z(M)$, then $M$ is C. R. 

Corollary 27: Let $d_1$ and $d_2$ are two nonzero \( R \cdot D \)'s of \( M \) (\( M \) is a two torsion free), \( A \) is a nonzero semigroup ideals of \( M \), if $d_1(\mathcal{A}) d_2(\mathcal{A}) \subseteq Z(M)$, then $M$ is C. R. 

Corollary 28: Let $d_1$ and $d_2$ are two nonzero \( R \cdot n \cdot D \)'s of \( M \) (\( M \) is a two torsion free), if $d_1(M, M, ..., M) d_2(M, M, ..., M) \subseteq Z(M)$, then $M$ is C. R. 

Corollary 29: Let $d_1$ and $d_2$ are two \( R \cdot D \)'s of \( M \) (\( M \) is a two torsion free), if $d_1(M) d_2(M) \subseteq Z(M)$, then $M$ is C. R. 

Theorem 7: Let $d$ be a nonzero \( R \cdot n \cdot D \) of \( M \), where \( M \) is a two torsion free, $A_1, A_2, ..., A_n$ are nonzero semigroup ideals of \( M \), if $d(u_1, u_2, ..., (u_j, v_j), ..., u_n) \in Z(M)$ for any $u_j \in A_j$, $u_2 \in A_2$, ..., $u_j \in A_j$, $u_n \in A_n$, then $M$ is a C. R. 

Proof: By assumption 
\[
d(u_1, u_2, ..., (u_j, v_j), ..., u_n) \in Z(M) \quad \text{for any } u_j \in A_j, u_2 \in A_2, ..., u_j \in A_j, u_n \in A_n.
\] 
Therefore, 
\[
d(u_1, u_2, ..., (s u_j, s v_j), ..., u_n) = d(u_1, u_2, ..., (u_j, v_j), ..., u_n) + d(u_1, u_2, ..., (u_j, v_j), ..., u_n) s \in Z(M)
\] 
for any $u_j \in A_j, u_2 \in A_2, ..., u_j \in A_j, u_n \in A_n$, $s \in M$. Replace $s$ by $(u_j, v_j)$ in last equation to get 
\[
d(u_1, u_2, ..., (u_j, v_j), ..., u_n) (2(u_j, v_j)) \in Z(M)
\] 
for any $u_j \in A_j, u_2 \in A_2, ..., u_j \in A_j, u_n \in A_n$, $s \in M$. Using Lemma 1(ii) implies 
\[
d(u_1, u_2, ..., (u_j, v_j), ..., u_n) = 0 \text{ or } (2(u_j, v_j)) \in Z(M)
\] 
for any $u_j \in A_j, u_2 \in A_2, ..., u_j \in A_j, u_n \in A_n$. If there is $u_j, v_j \in A_j$ such that 
\[
d(u_1, u_2, ..., (u_j, v_j), ..., u_n) = 0 \text{ for any } u_j \in A_j, u_2 \in A_2, ..., u_n \in A_n,
\] 
so $(u_j, v_j) \in Z(M)$, according to Lemma 6. 

Return to the hypothesis: for any $u_1 \in A_1, u_2 \in A_2, ..., u_n \in A_n$, $s \in M$. 
\[
d(u_1, u_2, ..., (s u_j, s v_j), ..., u_n) = d(u_1, u_2, ..., (u_j, v_j), ..., u_n) s \in Z(M)
\] 
for any $u_1, u_2 \in A_1, u_2, v_2 \in A_2, ..., u_j \in A_j, u_n \in A_n$. Using Lemma 1(ii) in last result forces 
\[
d(u_1, u_2, ..., s, ..., u_n) \in Z(M) \text{ or } (u_j, v_j) = 0, \text{ if } d(u_1, u_2, ..., s, ..., u_n) \in Z(M), (u_j, v_j) = 0, \text{ if } d(u_1, u_2, ..., s, ..., u_n) \in Z(M)
\] 
replace $s$ by $2s(u_j, v_j)$. last expression can be written as: 
\[
d(u_1, u_2, ..., s, ..., u_n) (2(u_j, v_j)) \in Z(M) \quad \text{or } (u_j, v_j) = 0, \text{ which conclude that } d(u_1, u_2, ..., s, ..., u_n) (2(u_j, v_j)) \in Z(M), \text{ thus } 2(u_j, v_j) \in Z(M) \text{ according to Lemma 1(ii) and Lemma 6.}
\] 
Therefore, Eq.8 becomes $2(u_j, v_j) \in Z(M)$ for any $u_j, v_j \in A_j$, it follows $2(s u_j, s v_j) = s(2(u_j, v_j)) \in Z(M)$ for any $u_j, v_j \in A_j$, $s \in M$. Lemma 1(i) and two torsion freeness ensures that $(u_j, v_j) = 0$ for any $u_j, v_j \in A_j$, or $s \in Z(M)$ for any $s \in M$. Therefore $M$ is C. R. by Lemma 4, Lemma 9 and Lemma 2(iii). 

Corollary 30: Let $d$ be a nonzero \( R \cdot n \cdot D \) of \( M \), where \( M \) is a two torsion free, $A$ is a nonzero semigroup ideals of \( M \), if $d(u_1, u_2, ..., (u_j, v_j), ..., u_n) \in Z(M)$ for any $u_1, u_2, ..., u_j, v_j, ..., u_n \in A$, then $M$ is a C. R. 

Corollary 31: Let $d$ be a nonzero \( R \cdot D \) of \( M \), where \( M \) is a two torsion free, $A$ is a nonzero semigroup ideals of \( M \), if $d((u_j, v_j)) \in Z(M)$ for any $u_j, v_j \in A$, then $M$ is a C. R. 

Corollary 32: Let $d$ be a nonzero \( R \cdot n \cdot D \) of \( M \), where \( M \) is a two torsion free, $A_1, A_2, ..., A_n$ are nonzero semigroup ideals of \( M \), if $d((u_j, v_j)) \in Z(M)$ for any $u_j, v_j \in A_j$, $u_2 \in A_2, ..., u_n \in A_n$, then $M$ is a C. R. 

Corollary 33: Let $d$ be a nonzero \( R \cdot D \) of \( M \), where \( M \) is a two torsion free, if $d((s, t)) \in Z(M)$ for any $s, t \in M$, then $M$ is a C. R. 

Theorem 8: Let $d$ be a nonzero \( R \cdot n \cdot D \) of \( M \), where \( M \) is a two torsion free, $A_1, A_2, ..., A_n$ are nonzero semigroup ideals of \( M \), if $d((u_j, v_j), (u_2, v_2), ..., (u_n, v_n)) \in Z(M)$ for any $u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n \in A_n$, $v_n \in A_n$, then $M$ is a C. R. 

Proof: By assumption: for any $u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_n \in A_n$. 
\[
d((u_1, v_1), (u_2, v_2), ..., (s u_j, s v_j), ..., (u_n, v_n)) = d((u_j, v_j), (u_2, v_2), ..., s(u_j, v_j), ..., (u_n, v_n))
\] 
\[
=d((u_j, v_j), (u_2, v_2), ..., s, ..., (u_n, v_n))(u_j, v_j)
\] 
\[
+d((u_1, v_1), (u_2, v_2), ..., (u_j, v_j), ..., (u_n, v_n)) s \in Z(M)
\] 
for any $u_1, v_1 \in A_1, u_2, v_2 \in A_2, ..., u_j \in A_j, v_j \in A_j, v_n \in A_n$. 


for any \( u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n \), \( s \in \mathcal{M} \).

Replace \( s \) by \((u_j, v_j)\) in last equation to get:

\[
d\left((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)\right)
\]

\[
(2(u_j, v_j)) \in \mathcal{Z}(\mathcal{M})
\]

For any

\[
u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n
\]

Using Lemma 1(ii) implies

\[
d\left((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)\right) = 0
\]

or \( 2(u_j, v_j) \in \mathcal{Z}(\mathcal{M}) \)

for any

\[
u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n
\]

If there is \( u_0, v_0 \in \mathcal{A}_j \) such that

\[
d\left((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)\right) = 0
\]

for any \( u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n \).

Return to the hypothesis: for any

\[
u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_n, v_n \in \mathcal{A}_n
\]

\[
d\left((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)\right) = 0
\]

or \( 2(u_j, v_j) \in \mathcal{Z}(\mathcal{M}) \)

for any

\[
u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n
\]

then \( \mathcal{M} \) is a \( \mathcal{C.R} \) because of Theorem 2.

If \((u_0, v_0) \in \mathcal{Z}(\mathcal{M})\), then using Eq. 11 leads to

\[
d\left((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots, (u_n, v_n)\right) = 0
\]

for any \( u_1, v_1 \in \mathcal{A}_1, u_2, v_2 \in \mathcal{A}_2, \ldots, u_j, v_j \in \mathcal{A}_j, \ldots, u_n, v_n \in \mathcal{A}_n \),

then \( \mathcal{M} \) is a \( \mathcal{C.R} \) because of Lemma 1(iii).


