Best approximation in b-modular spaces

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Abstract

In this paper, some basic notions and facts in the b-modular space similar to those in the modular spaces as a type of generalization are given. For example, concepts of convergence, best approximate, uniformly convexity etc. And then, two results about relation between semi compactness and approximation are proved which are used to prove a theorem on the existence of best approximation for a semi-compact subset of b-modular space.

Keywords: b-Modular vector spaces, Best approximation, Fixed point, Semi-compactness, Set valued mapping.

Introduction

The primary problems of best approximation theory from Ky Fan’s point of view a convexity advantage which require introducing a mapping with some hypotheses. In this article, the focus was on Ky Fan’s type best approximation: Let $\Omega$ be convex compact subset of a normed linear space $\Gamma$, $f$ is a continuous function on $\Omega$ and $u$ is an element of $\Gamma$; that approximately to $u$ from the elements in $\Omega$ would be a vector $v \in \Omega$ such that $\|v - f(v)\| = d(f(v),\Omega)^1$, where $d$ is a metric distance induced by norm.

The value of Ky Fan’s Theorem1 is due to its use in elicitation many fixed-point theorems as a corollary depending on weaker assumptions in many fields of nonlinear analysis1, This theorem has been of great importance in nonlinear analysis, approximation theory, fixed point theory and variational inequalities. The result is equivalent to the well-known topological fixed point theorem due to Brouwer introduced by Kanster, Knratowski, and Mazurkiewicz. They produced a very important result depending on Sperner’s lemma, presently, it is known as the (KKM-map) principle. This principle was employed to give a simplified proof of Brouwer’s theorem. An extension of KKM-map theorem was presented by Ky Fan in topological vector spaces and gave several interesting applications, specially, in fixed point theory and best approximation theory. Fixed point theorems have been used at many places in approximation theory. Later on, many results were developed using fixed point theorem to prove the existences of best approximation. By H. Kaneko sufficient conditions for the existence of a coincidence point of continuous multivalued mappings are derived in $p$-normed spaces (0 < $p$ < 1). As applications, some results on the set of best approximation for this class of mappings are obtained. A. Latif obtained the coincidence point theorem for Banach spaces or thesis sources (for more details see 1-4).

The aim of this article is to establish some relations among approximate compact set, compact set and proximinal set in modular spaces. A first
Various results in modular spaces a proximinal set, Chebysev set are proven also, see Example 4 Example 3: Let \( \theta \) be a non-negative convex Orlicz function such that \( \theta(0) = 0 \). The Orlicz space is a modular function space generated by

\[
\rho(f) = \int_R \theta(|f(t)|) \, dm(t)
\]

Example 2: Let \( \mu \) be a \( \theta \)-finite measure. \( F = \{ \tau: R \rightarrow R \mid \tau \text{ is measure preserving transformations}, \mu_\tau(E) = \mu(\tau^{-1}(E)) \} \). The group \( F \) is a modular function space generated by \( \rho(f) = \sup_{\tau \in F} \int_R |f(t)|^p \, d\mu_\tau(t) \) is a Musielak-Orlicz. Note, throughout the work, \( R \) is a symbol of real numbers.

Example 3: Let \( \mu \) be a \( \sigma \)-finite measure. \( F = \{ \tau: R \rightarrow R \mid \tau \text{ is measure preserving transformations}, \mu_\tau(E) = \mu(\tau^{-1}(E)) \} \). The group \( F \) is a modular function space with

\[
\rho(f) = \sup_{\tau \in F} \int_R |f(t)|^p \, d\mu_\tau(t)
\]

is a Lorentz \( \rho \)-space.

Example 4: Let \( \varphi \) be an Orlicz function. \( F = \{ \tau: R \rightarrow R \mid \tau \text{ is measure preserving transformations and} \mu_\tau(E) = \mu(\tau^{-1}(E)) \} \). The group \( F \) is a modular function space generated by

\[
\rho(f) = \sup_{\tau \in F} \int_R \varphi(|f(t)|) \, d\mu_\tau(t)
\]

is a Orlicz-Lorentz space.

In 2017, the best approximation modular spaces have been defined by Abed and results about proximinal set, Chebysev set are proven also, see\(^8\). Various results in modular spaces and other related spaces about fixed point problem can be seen in Turkoglu and Nesrin\(^9\), Albundii\(^10\), Abdul Jabbar and Abed\(^11\), Ahmed\(^12\), Mohammed and Abed\(^13\), Pathak and Beg\(^14\), Abed and Abdul Jabbar\(^15\), Ege and, Alac\(^16\) also Abed and Salman\(^19\).

**Preliminaries**

A generalization of modular space should be mentioned here as defined as modular b-metric space\(^17\).

**Definition 1** Let \( \Gamma \) be a vector space over \( F(= R \ or \ \mathbb{C}) \), a function \( \zeta: \Gamma \rightarrow [0, \infty] \) is called \( b \)-modular if

(i) \( \zeta(v^A) = 0 \) if and only if \( v^A = 0 \). \( \zeta(\alpha v^A) = \alpha \zeta(v^A), \ \alpha \in F \) and \( |\alpha| = 1 \forall v^A \in \Gamma \).

(ii) \( \zeta(\alpha v^A + \beta u^A) \leq b(\zeta(v^A) + \zeta(u^A)) \) if \( \alpha, \beta \geq 0 \), for all \( v^A, u^A \in \Gamma, b \geq 1 \). If (iii) replaced by

(iii) \( \zeta(\alpha v^A + \beta u^A) \leq b(\alpha \zeta(v^A) + \beta \zeta(u^A)) \), for \( \alpha, \beta \geq 0, \alpha + \beta = 1 \), for all \( v^A, u^A \in \Gamma, b \geq 1 \). Then \( \Gamma \) modular \( \zeta \) is called convex \( b \)-modular.

When \( b=1 \), it will be convex modular. Similar to \(^*\) set the following.

**Definition 2** A function \( \zeta \) defines an identical \( b \)-modular space \( I_\zeta \), as follows

\( I_\zeta = \{ v^A \in \Gamma; \zeta(\alpha v^A) \rightarrow 0 \text{ whenever } \alpha \rightarrow 0 \} \).

**Example 5** The space \( l_p = \{ u = \{ u_n^A \} \in R, \sum |u_n^A|^p < \infty \}, 0 < p < 1 \), with modular \( \zeta(u) = (\sum |u_n^A|^p)^{1/p} \) is \( b \)-modular space with \( b=2 \) by similar details in\(^4\).

**Remark 1**

i) If \( u^A = 0 \) then \( \zeta(\alpha v^A) = \zeta(\frac{\alpha}{\beta} v^A) \leq \zeta(\beta v^A), \) for all \( \alpha, \beta \in F, 0 < \alpha < \beta \), by condition (iii) above and by similar reasoning in \(^6\) and this shows that \( \zeta \) is increasing function.

ii) The family of all \( \zeta \)-balls in the space \( I_\zeta \) gives a topology.

In the sense in \(^13\), the following definition is stated.

**Definition 3** The distance between \( v^A \in I_\zeta \) and \( B \subseteq I_\zeta \) is

\( D_\zeta(v^A, B) = \inf \{ \zeta(\zeta(v^A - u^A); u^A \in B) \} \).

**Definition 4** Let \( I_\zeta \) be a \( b \)-modular space

(a) A sequence \( \{ v_n^A \} \subseteq I_\zeta \) converges to \( v^A \in I_\zeta \) if \( \zeta(v_n^A - v^A) \rightarrow 0 \) as \( n \rightarrow \infty \). It is called \( \zeta \)-convergent to \( v^A \) write \( v_n^A \xrightarrow{\zeta} v^A \)
Reformed the concept of upper semi-continuous mapping (shortly, u.s.c.).

**Definition 6** A set-valued mapping $F$ is u.s.c., if the set $\{u^A \in \Gamma: F(x) \cap B \neq \emptyset\}$ is closed whenever $B$ is closed subset of $N_p$. In natural way, the following are defined, see [14].

**Definition 7** If $\Omega$ is a subset of $\Gamma$, then:

(i) $\Omega$ is called proximinal if for all $v^A \in \Gamma$, there exists a $u^A \in \Omega$ such that $\zeta|\{v^A - u^A\}| = D_\zeta(v^A, \Omega)$.

(ii) $\Omega$ is called Chebysev if for each $v^A \in \Gamma$, there is a unique element $u^A \in \Gamma$ such that $\zeta|\{v^A - u^A\}| = D_\zeta(v^A, \Omega)$.

**Definition 8** A collection of all best approximation of $v^A \in \Gamma$ by $\Omega$ is

$$P_\Omega(v^A) = \{u^A \in \Omega: \zeta(v^A - u^A) = D_\zeta(v^A, \Omega)\}$$

and $P_\Omega: M \rightarrow 2^B$ is said to the metric projection on $\Gamma$, where $2^B$ is the class of all nonempty subset of $\Omega$.

**Proposition 1** If $\Omega$ is closed convex subset of a uniformly convex space $\Gamma$, then it is semi-compact.

Proof: Suppose $\Gamma \in \Omega$ uniformly convex, $\Omega = \Gamma$, $\Omega$ is convex and closed and $u^A \in \Gamma$ and $u^A \neq \Omega$ such that $\zeta(u^A - u^A) \rightarrow D_\zeta(u^A, \Omega)$. Then sup $\zeta(u^A) < \infty$. The closeness and convexity of $\Omega$ implies there is $u^0 \in \Omega$ and a sequence $u^A_n \subseteq \Omega$ such that $u^A_n \rightarrow u^0$. As $\limsup \zeta(u^A_n - u^A) < \infty$. So

$$\zeta(u^0_n - u^A) \leq \limsup \zeta(u^A_n - u^A) = D_\zeta(u^A, \Omega)$$

that is $\zeta(u^0_n - u^A) = D_\zeta(u^A, \Omega)$. By definition of $u^A_n$, getting $u^A_n \rightarrow D_\zeta(u^A, \Omega) = u^A - u^A$.

Since $\Gamma$ is a uniformly convex, then getting $u^A_n \rightarrow u^A - u^A$, that is then $u^A_n \rightarrow u^A \in A$. Then $A$ is a semi compact.

**Theorem 1** If $\Omega$ is semi compact subset of $\Gamma$, then $\Omega$ is a proximinal and closed

**Proof:** Let $v^A \in \Gamma$. by definition of $D_\zeta(v^A, \Omega)$, from the set of the numbers $\zeta(v^A - u^A)$, we can construct a sequence $\zeta(v^A - u^A)$ such that $\limsup \zeta(v^A - u^A) = D_\zeta(v^A, \Omega)$.

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$$\zeta(u^0_n - u^A) \leq \limsup \zeta(u^A_n - u^A) = D_\zeta(u^A, \Omega)$$

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since $\Omega$ is a semi-compact. Then from $\langle u_n^d \rangle$, there is a subsequence converging to a point $u_0^d \in \Omega$. Hence, by the continuity of $\zeta$ getting

$$\zeta(v^d - u_0^d) = \lim_{n \to \infty} \zeta(v^d - u_n^d) = \lim_{n \to \infty} \zeta(v^d - u_n^d) = D_\zeta(v^d, \Omega)$$

when $u_n^d \in P_d(v^d)$, the proof of proximinal is complete. Finally, for an accumulation point $v^d$ of $\Omega$, then $\exists u^d \in \Omega$ such that $(v^d - u^d) = D_\zeta(v^d, \Omega) \Rightarrow 0$, so $v^d \in \Omega$, and $\Omega$ is closed set.

Returning to Remark 2, to show the opposite fails, consider $I_\Omega = i_2(R)$ uniformly convex complete space with convex modular $\psi(x) = \sqrt{\sum_1^\infty |x_t|^2}$, $\Omega = \{v \in I_\Omega : \zeta(v) \leq r\}$, $r > 0$, defined by $u_{\{1\}} = 0$ and $u_{n} = \left(1, \frac{1}{n}, 0, ... , 0, 1, 0, ...ight)$, $n \geq 2$ is proximal but not semi compact.

**Theorem 2** Let $\emptyset \neq \Omega \subseteq I_\Omega$ and $\Omega$ be a semi-compact. If $\zeta(u^A) < \infty$ for each $u$. Then $P_d$ maps $I_\Omega$ into $CB(\Omega)$ is u.s.c., where $CB(\Omega) = \{\emptyset \neq \Sigma \in \Omega, \Sigma$ is closed and bounded}. 

**Proof:** By Theorem 1, $\Omega$ is proximinal set. Then $P_d(v^d)$ is non-empty for each $v^d \in I_\Omega$. So, $P_d(v^d)$ is closed and bounded (in the sense of 9). Thus $P_d(v^d)$ maps $I_\Omega$ into $CB(\Omega)$. Now, let $\Sigma \in CB(\Omega)$ and define the set

$$B = \{v^d \in I_\Omega : P_d(v^d) \cap \Sigma = \emptyset\}$$

To complete the proof, it is enough to prove that $B$ is closed. Let $\langle u_n^d \rangle$ be a sequence in $B$, converging to $v^d \in I_\Omega$. Assume $\{u_n^d\} \subseteq B$, then there is a sequence $\langle u_n^d \rangle \subseteq \Omega$ such that $\langle u_n^d \rangle \subseteq P_d(v_n^d)$ for $\emptyset \neq \Sigma$ (where $n = 1, 2, ...$). By $\langle u_n^d \rangle \subseteq P_d(v_n^d)$, $(n = 1, 2, ...$). You have

$$\lim_{n \to \infty} D_\zeta(v_n^d, \Omega) = \lim_{n \to \infty} \zeta(v_n^d - u_n^d)$$

or $D_\zeta(v_n^d, \Omega) = \lim_{n \to \infty} \zeta(v_n^d - u_n^d)$

$$\leq b \lim_{n \to \infty} \zeta(v_n^d - u_n^d) + \lim_{n \to \infty} \zeta(v_n^d - u_n^d)$$

Accordingly,

$$\lim_{n \to \infty} \zeta(v_n^d - u_n^d) = \lim_{n \to \infty} \zeta(v_n^d - u_n^d)$$

Thus $\lim_{n \to \infty} \zeta(v_n^d - u_n^d) = D_\zeta(v_n^d, \Omega)$. Consequently, the semi compactness provides a subsequence $\langle u_n^d \rangle$ of $\langle u_n^d \rangle$ converging to $u_\Omega^d$, so there is a subsequence $\langle v_n^d \rangle$ of $\langle u_n^d \rangle$. Now, since $u_\Omega^d \in \Omega$, then

$$D_\zeta(v_n^d, \Omega) \leq \zeta(v_n^d - u_n^d) \leq b[\zeta(v_n^d - u_n^d) + \zeta(u_n^d - u_\Omega^d)]$$

$$\leq b^2 \zeta(v_n^d - u_n^d) + b^2 D_\zeta(v_n^d, \Omega)$$

for $k \to \infty$, $\zeta(v_n^d - u_n^d) = D_\zeta(v_n^d, \Omega)$, that is $u_\Omega^d \in P_d(v_n^d)$. Also, since $\Sigma$ is a closed and $\langle u_n^d \rangle \subseteq \Sigma$, $\lim_{k \to \infty} u_n^d = u_\Omega^d$ have $u_\Omega^d \in P_d(v_n^d) \cap \Sigma$. Thus, the proof is complete.

**Theorem 3** Suppose that $\Omega$ Theorem 2, and $P_d : I_\Omega \to 2^{\Omega}$ is the metric projection of $I_\Omega$ onto $\Omega$. If $\Sigma$ compact subset of $\Gamma$ then $P_d(\Sigma) = \cup \left(P_d(v^d) : v^d \in \Sigma \right)$ is compact.

**Proof:** Assume $\langle u_n^d \rangle$ is a sequence in $P_d(\Sigma)$. So, there is a sequence $\langle u_n^d \rangle \subseteq \Sigma$ such that for each $u_n^d \in P_d(v_n^d)$, that is $\zeta(v_n^d - u_n^d) = D_\zeta(v_n^d, \Omega)$.

Since $\Sigma$ is compact, then it may assume that there is a $v^d \in \Sigma$ with $u_n^d \to v^d$ and

$$D_\zeta(v^d, \Omega) \leq \zeta(v^d - u_n^d) \leq b[\zeta(v^d - u_n^d) + D_\zeta(v_n^d, \Omega)]$$

Therefore,

$$\lim_{n \to \infty} D_\zeta(v_n^d, \Omega) = \lim_{n \to \infty} \zeta(v_n^d - u_n^d)$$

$$\leq \lim_{n \to \infty} \zeta(v_n^d - v_n^d)$$

$$\leq \lim_{n \to \infty} D_\zeta(v_n^d, \Omega)$$

$$D_\zeta(v^d, \Omega) \leq \lim_{n \to \infty} \zeta(v^d - u_n^d) \leq D_\zeta(v^d, \Omega) \Rightarrow D_\zeta(v^d, \Omega) = \lim_{n \to \infty} \zeta(v^d - u_n^d)$$

By semi compactness of $\Omega$ and $\langle u_n^d \rangle \subseteq P_d(C) \subseteq \Omega$ , the above steps imply to the existences of $u_n^d \in \Omega$ and subsequence $\langle u_n^d \rangle$ of $\langle u_n^d \rangle$ with $u_n^d \to u^d$. This prove that $P_d(\Sigma)$ is compact.

**Conclusion**

This paper includes many basic concepts and facts in the convex modular vector space, which were employed to obtain some results, such as, a semi-compact subset of modular space is closed proximinal which means that, $P_d v^d \neq \emptyset$ for each $v^d \in \Omega$ and images of $P_d$ is compact. In the future, you can use the results to obtain some applications in other fields, such as control.
Open Problem

It possible to combine the set $P_{\mathcal{U}}(\sum)$ into work in 11, 12 and 16 to posing the following question: Could the limit of convergence iterative sequences in 11, 12 and 16 be an invariant best approximation?

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Authors’ Declaration

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are ours. Furthermore, any Figures and images, that are not ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors’ Contribution Statement

This work was carried out in collaboration between all authors. S S, the owner of the research idea and she reviewed and processed the work. S N, and H A, written and proved the results. All authors read and approved the final manuscript.

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**احصائيات**

- المفاهيم الأساسية في الفضاءات المعيارية: b-
- أفضل تقريب في فضاءات b-

**المختصرة**

في البداية، نعطي بعض المفاهيم والحقائق الأساسية في الفضاءات b- المعايير. ندعم هذه المفاهيم من خلال العددينما ونتعلق بال пряحا. نستخدم هذه الفضاءات في الفضاءات غير متراصة وت答え إلى ذلك، ثم نرتب على القيم للعلاقة بين شبه التراص والتقارب، والتي استخدمت لإثبات مبرهنة حول وجود أفضل تقريب لمجموعة جزئية شبه متراصة من الفضاء b- المعياري.

**الكلمات المفتاحية:** b- فضاء المعياري، أفضل تقريب، النقطة الصامدة، شبه التراص، تطبيق متعدد القيم.