On Faintly $\theta$- Semi-Continuous and Faintly $\delta$-Semi-Continuous Functions

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Received 11/01/2023, Revised 08/05/2023, Accepted 10/05/2023, Published Online First 20/08/2023

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Abstract

Faintly continuous (FC) functions, entitled faintly $\theta S$-continuous and faintly $\delta S$-continuous functions have been introduced and investigated via a $\theta$-open and $\delta$-open sets. Several characterizations and properties of faintly $\theta S$-continuous and faintly $\delta S$-Continuous functions were obtained. In addition, relationships between faintly $\theta S$- Continuous and faintly $\delta S$- continuous function and other forms of FC function were investigated. Also, it is shown that every faintly $\theta$ $S$-continuous is weakly $\theta S$-continuous. The Convers is shown to be satisfied only if the co-domain of the function is almost regular.

Keywords: Faintly $\theta S$-continuous, Faintly $\delta S$-continuous, $\theta$-open, $\delta$-open, Faint continuity.

Introduction

Faint continuity is a property weaker form of Continuous functions. Throughout this paper, since the introduction of FC functions by Long and Herrington1, various weak and strong forms of FC functions were studied. Many authors defined and introduced a generalization form of open sets and weak and strong forms of semi-open sets see in 2-6. The concept FC functions have the attention of many authors see for example 7. First, a point $\chi \in X$ is called an $\theta$ -Cluster point of $E \subseteq X$ if $E$ non-trivially intersects the closure of each open set containing $\chi$ in $X$. All $\theta$ -Cluster set points of some set is defined to be the $\theta$ closure of that set and its written as $\text{Cl} \theta (E)$. A subset that contains all its $\theta$ -Cluster points (i.e., $E = \text{Cl} \theta (E)$), is $\theta$-closed, and its complement is $\theta$ -open. Equivalently, $E$ is $\theta$-open if it has a closed neighborhood of each of its points. Another equivalent definition is that if for all $\chi \in E$, an open set $O$ exists with the property that $\chi \in O \cap \text{Cl}(O) \subseteq E$. The collection, $T \theta$, of $\theta$ -open subsets in $X$ forms a topology on $X$. The $\text{int} \theta (E)$ is the largest $\theta$-open subset of $E$. A $\delta$ -Cluster point $\chi \in X$ of $E \subseteq X$ is a point s.t. for every open set $U \ni \chi$ its (int $(\text{Cl}(O) \cap E \neq \emptyset$). The $\delta$-closure of a set $E$, $\text{Cl} \delta (E)$, is the set of all $\delta$-

Cluster points of that set. A $\delta$-closed is one which equals its $\delta$-closure. A $\delta$-open set is one whose complement is $\delta$-closed. Equivalently, $E$ is $\delta$-open if for all $\chi \in E$ there is a regular-open (r-open) subset of $E$ containing $\chi$. The collection of all $\delta$-open subsets of $X$ is a topology on $X$ denoted by $T \delta$. A subset $E$ in $X$ is proclaimed semi-open denoted by (SO) if $\exists$ an open set $Q$ s.t $Q \subseteq E \subseteq \text{Cl}(Q)^{3}$.

Preliminaries:

The terms ($X$, $\tau$) and ($Y$, $\sigma$) pertain to topological spaces where there are no underlying separation axioms. The closure, interior and the complement of a set $E$ are denoted respectively by Cl (E), int(E) and $A^{c}$. A point $\chi \in X$ is called the $\theta$-Cluster point (respectively $\delta$-Cluster point) of the set $E$ if for every open subset $Q$ of $X$ s.t. $\chi \in Q$, $E \cap \text{Cl}(Q) \neq \emptyset$ (respectively $E \cap \text{int}(\text{Cl}(Q)) \neq \emptyset$). The set of all $\theta$-Cluster points (respectively $\delta$-Cluster points) of a set, $E$, is said to be the $\theta$-closure of $E$ denoted by $\text{Cl} \theta (E)$ (respectively $\delta$-closure denoted by $\text{Cl} \delta (E)$). Also $\theta$-closed (respectively $\delta$-closed) set is one which equals its respective closure. A $\theta$ -open set (respectively $\delta$-open) is one whose complement is $\theta$
-closed (respectively δ-closed)\(^9\). A set E in a topological space \((X, \tau)\) is \(\theta\)-semi-open if \(\exists\ a\ \theta\ -open\ subset\ Q\ of\ X\ s.t\ Q \subset \text{Cl}(Q)\). Equivalently, if \(E \subset \text{Cl}(\text{int}(\theta)(E))\) \(^10\). A set is \(\theta\)-semi closed if its complement is \(\theta\)-semi open. A set \(E\) in a topological space \((X, \tau)\) is \(\delta\)-semi-open if \(\exists\ \delta\ -open\ subset\ Q\ of\ X\ s.t\ Q \subset \text{Cl}(Q)\) \(^11\). Equivalently, if \(E \subset \text{Cl}(\text{int}(\delta)(E))\).

\(\delta\)-semi closed set is one whose complement is \(\delta\)-semi open, its denote by \(\delta s(\chi)\) for the collection consisting of all \(\delta\)-semi open sets in a space \(X\). A mapping \(f\colon (X, \tau) \to (Y, \sigma)\) is called faintly Continuous (FC) (respectively faintly semi-Continuous (FSC)) if \(\forall\ \chi \in X\) and every \(\delta\)-open subset \(Q\ of\ Y\), \(f(\chi) \in Q\), there is an open subset (respectively semi-open), \(O\), of \(X\), containing \(\chi\ s.t\ f(O) \subset Q\). Equivalently, \(f\) is FC (respectively FSC) if the pre-image of each \(\theta\)-open set is open (semi-open) set. A regular-open set \(Q\) is one s.t. \(Q = \text{int}(\text{Cl}(Q))\). A regular closed (r-closed) is one whose complement is r-open. Equivalently, \(F\) is r-closed set if \(\text{Cl}(\text{int}(F)) = F\). The point \(\chi \in X\) is a \(\delta\)-Cluster point of some set \(E\ if\ E \cap O \neq \emptyset\ \forall\ \delta\)-semi open subset \(O\ of\ X,\ \chi \in O\).

\(\delta\)-semi closure of a subset \(E\) (which denotes \(\text{Cl}(\delta)(E)\)) is the set consisting of its \(\delta\)-semi-Cluster points. The family consisting of \(\delta\)-semi open sets (respectively \(\delta\)-semi closed) will be denoted by \(\delta \text{SO}(X, \tau)\) (respectively \(\delta \text{SC}(X, \tau)\)). Then, \(\text{Cl}(\delta)(E) = \cap\{F: E \subset F, F is \(\delta\)-semi closed\}\). The concept of \(\theta\)-semi closure of some subset \(E\) of a space \(X\), denoted \(\text{cl}(\theta)(E)\), is the set of all \(\chi \in X\ s.t\ Cl(O)\cap \chi \neq \emptyset\) for all semi open subset \(O\ of\ X\ s.t.\ \chi \in \text{Cl}(O)\).

### Results and Discussion

#### Characterization of faintly \(\theta\)-semincontinuity:

**Definition 1:** \(^12\) \(f\colon (X, \tau) \to (Y, \sigma)\) is faintly \(\theta\)-semi-Continuous (\(\theta\)-S-continuous) if for all point \(\chi \in X\) and every \(\theta\)-open set \(Q\ of\ Y\), \(f(\chi) \in Q\), there is an \(\theta\)-semi open set \(O\ of\ X\), \(\chi \in O\ s.t\ f(O) \subset Q\). Equivalently, \(f\) is SC (respectively S-SC) if \(\forall\ \chi \in X\) and \(\forall\ \theta\)-open set \(Q\), \(f(\chi) \subset Q\). The concept of \(\theta\)-semi closure of some subset \(E\) of a space \(X\), denoted \(\text{Cl}(\theta)(E)\), is the set of all \(\chi \in X\ s.t\ Cl(O)\cap \chi \neq \emptyset\) for all semi open subset \(O\ of\ X\ s.t.\ \chi \in \text{Cl}(O)\).

**Theorem 1:**

Given a mapping \(f\colon (X, \tau) \to (Y, \sigma)\), such that \(Y\) is an almost regular space, then the equivalence of the following statements can be established:

i. The mapping \(f\) is faintly \(\theta\)-S-continuous.

ii. The pre-image \(f^{-1}(Q)\) is a \(\theta\)-semi open subset of \(X\) whenever \(Q\) is r-open subset of \(Y\).

iii. The pre-image \(f^{-1}(F)\) is a \(\theta\)-semi closed set of \(X\) whenever \(F\) is r-closed is a set in \(Y\).

iv. If \(E \subset X\) then, \(f(\text{cl} \theta(E)) \subset \text{Cl} \theta(f(E))\).

v. If \(B \subset X\) \(f^{-1}(B) \subset f^{-1}(\text{Cl}(\theta)(B))\).

vi. The pre-image \(f^{-1}(F)\) is a \(\theta\)-semi closed for all \(\theta\)-closed set \(F\) of \(Y\).

vii. For every \(\theta\)-open subset \(Q\ of\ Y,\ f^{-1}(Q)\ is\ a\ \theta\)-semi open subset of \(X\).

**Proof:**

(i \(\Rightarrow\) ii) Assume that \(Q\ is\ a r-open\ in\ \ Y, in addtion let \(\chi \in f^{-1}(Q)\), then \(Q = \text{int}(\text{Cl}(Q))\) and \(f(\chi) \subset Q, openness\ of\ Q, implies\ \theta - the openness\ in\ Y\ [because\ Y\ is\ almost\ regular]\ and by application of part (i) and the definition 1, \(\exists a \theta\ -semi\ open\ set\ O\ of\ X\ s.t\ f(O) \subset Q\). Therefore, \(\chi \in O\ \ s.t\ f(O_{C}) \subset f^{-1}(f(O_{C})) \subset f^{-1}(Q)\) and \(\exists a \theta\ -open\ set\ \ W\ s.t\ W \cap O = \emptyset\) since \(O_{C}\ is\ a\ \theta\ -semi\ open\ set. Now, suppose that \(W_{o} = \emptyset\ \ s.t\ f(O) \subset W_{o}\). As \(\cup_{\chi \in f^{-1}(Q)} \text{Cl}(W_{\chi}) \subset \text{Cl}(W)\ then\ O = \cup_{\chi \in f^{-1}(Q)} (O_{C}) = f^{-1}(Q)\ and the proof is concluded.

(ii \(\Rightarrow\) iii) Suppose that \(F\ is\ r-closed\ in\ Y_{o}\ so\ \forall\ \chi \in f^{-1}(Q)\) is r-open in Y. BY (ii), \(f^{-1}(\text{Cl}(F))\ is\ a\ \theta\ -semi\ open\ subset\ of\ X. Since\ f^{-1}(\text{Cl}(F)) = f^{-1}(\text{Cl}(F))\ and \(f^{-1}(F)\ is\ then\ implied\ that\ f^{-1}(F)\ is\ a\ \theta\ -semi\ closed\ subset\ of\ X.\)

(iii \(\Rightarrow\) iv) Suppose that \(\chi \in \text{Cl}(\theta)(E)\ s.t\ f(\chi) \not\in \text{Cl}(\theta)(f(E)).\) Then, \(\exists\ a\ \theta\ -open\ set\ \ W_{o}\ s.t\ f(O) \subset W_{o}\). When taking the r-open set \(W_{o} = \text{int}\ (\text{Cl}(Q_{o})). Then\ Cl(W_{o}) = Cl(Q_{o}). Thus, \(f(O) \subset \text{Cl}(W_{o})\). BY part (iii), it got \(\chi \in \text{Cl}(f^{-1}(W_{o}))\ is\ \theta\ -semi\ closed\ and\ \subset \text{Cl}(f^{-1}(W_{o})). Thus, by the definition of \(\text{Cl}(\theta)(E)\), it got \(\chi \in X\ \ s.t\ f^{-1}(W_{o})\), a contradiction with \(f(\chi) \subset Q_{o}\ s.t\ f^{-1}(Cl(Q_{o})) = W_{o}.

(iv \(\Rightarrow\) v) Assume that \(E = f^{-1}(B) \subset X\). Then, by part (iv) it has \(f(\text{cl}(\theta)(E)) \subset \text{Cl}(\theta)(f(E)). Then, Cl(\theta)(f(E)) \subset Cl(\theta)(B), it follows that \text{Cl}(\theta)(E) \subset f^{-1}(\text{Cl}(\theta)(B)).\)
(vii) Suppose that \( x \in X \) and \( Q \) be an \( \theta \) - open subset of \( Y \), \( \exists \) disjoint open subsets \( O \) and \( Q \) of \( X \), s.t \( \chi \in O \) and \( f \in C_l(Q) \), consequently, \( f \) is faintly \( \theta \) - S-continuous.

Definition 4:
Suppose that \( f: X \rightarrow Y \) be a function, the function \( g: X \rightarrow X \times Y \) is called a graph function of \( f \) if \( g \) is defined by \( g(x) = (x, f(x)) \) for each \( x \in X \).

Theorem 5:
Given the graph map of \( f: X \rightarrow Y \) to be is faintly \( \theta \) S-continuous, then so is \( f \).

Proof: Assume that \( \chi \in X \), \( Q \) an \( \theta \) -open subset of \( Y \), \( f(\chi) \in Q \Rightarrow X \times Y \) is \( \theta \) -open subset of \( X \times Y \) [By Theorem 5 in'] containing \( g(\chi) = (\chi, f(\chi)) \). Since the graph map \( g: X \rightarrow X \times Y \) is faintly \( \theta \) S-continuous, there exists \( O \in \theta \) \( SO(X) \) containing \( X \) s.t \( g(O) \subset X \times Q \), then \( f(O) \subset Q \). Hence, faint \( \theta \) -S-continuity of \( f \) is established.

Theorem 6:
Supposing \( f \) is faintly \( \theta \) S-continuous with \( Y \) is almost regular. For all \( \chi \in X \) and \( \theta \) -open subset \( Q \) in \( Y \) s.t \( f(\chi) \in Q \), \( \exists \) \( \theta \) -s-open subset \( O \) of \( X \) s.t \( f(O) \subset \text{int}(Cl(Q)) \).

Proof: If \( \chi \in X \) with \( f(\chi) \in Q \), where \( Q \) is a \( \theta \) -open subset in \( Y \), \( \exists \) \( \theta \) -open subset \( G \) in \( Y \) s.t \( f(\chi) \in G \subset \text{Cl}(G) \subset Q \subset \text{int}(Cl(Q)) \).

(\( \gamma \Rightarrow \gamma \)) Assume that \( \gamma \) is a \( \theta \) -Closed of \( Y \). \( \exists \) \( SO (F) \subset \text{Cl}(\theta) \) (F) = \( F \). Taking \( B = \text{F in part (v)} \), it got \( s \text{Cl}(\theta) (f^{-1}(F)) \subset f^{-1}(F) \). As \( f^{-1}(F) \subset s \text{Cl}(\theta) (f^{-1}(F)) \), then \( f^{-1}(F) \) is \( \theta \) -semi closed, it concludes \( f^{-1}(F) \) is \( \theta \) -semi closed subset of \( X \).

(vii) Let \( Q \) be \( \theta \) -open subset of \( Y \). Taking \( F = Q \in (vii) \) in part (vi) it got \( f^{-1}(Q) = \{f^{-1}(Q)\} = \theta \) -semi closed subset of \( X \), then \( f^{-1}(Q) \) is \( \theta \) -semi open subset of \( X \).

Theorem 3:
Supposing \( f \) is faintly \( \theta \) -S-continuous with \( Y \) being almost regular, then \( f \) is faintly \( \theta \) S-continuous.

Proof: Let \( \chi \in X \). Assume further that \( Q \) is \( \theta \) - open set of \( Y \), \( f(\chi) \in Q \). So, there exists \( r \)-open set \( W \) in \( Y \) s.t \( f(\chi) \in W \subset Cl(w) \subset Q \). By Theorem 1 in', since \( Y \) is almost regular, then each \( r \)-open set in \( Y \) is also \( \theta \) - open [by Theorem 3 in']. Now, by weak \( \theta \) -S-continuity of \( f \), \( \exists \theta \) -s-open subset \( O \), \( \chi \in O \subset (s) \subset Cl(w) \subset Q \Rightarrow f(O) \subset Q \), consequently, faint \( \theta \) -S-continuity of \( f \) is established.

Definition 3:
A space \( X \) is almost regular if whenever \( F \) r-closed subset in \( X \) with \( \chi \notin F \), \( \exists \) disjoint open subset \( O \) and \( Q \) of \( X \), s.t \( \chi \in O \) and \( f \subset F \subset C(f) \).

Theorem 4:
If \( f(\chi), \tau \rightarrow (Y, \sigma) \) is weakly \( \theta \) S-continuous, with \( Y \) being almost regular, then \( f \) is faintly \( \theta \) S-continuous.

Proof: Assume that \( \chi \in X \), \( Q \) be an \( \theta \) -open subset of \( Y \), \( f(\chi) \in Q \). There exists \( r \)-open \( W \) in \( Y \) s.t \( f(\chi) \in W \subset Cl(w) \subset Q \). By Theorem 1 in', since \( Y \) is almost regular, then each \( r \)-open set in \( Y \) is also \( \theta \) - open [by Theorem 3 in']. Now, by weak \( \theta \) -S-continuity of \( f \), \( \exists \theta \) -s-open subset \( O \), \( \chi \in O \subset (s) \subset Cl(w) \subset Q \Rightarrow f(O) \subset Q \), consequently, faint \( \theta \) -S-continuity of \( f \) is established.
[by Theorem 3 in\(^1\)] so, \(G\) is \(r\)-open, when \(Y\) is almost regular then \(G\) is \(\theta\)-open [by theorem 3 in\(^1\)], faint \(\theta\) \(S\)-continuity of \(f\) means that it can find a \(\theta\)-open set \(O\) in \(X\) s.t. \(\chi\in O\) and \(f(O)\subset G\subset Cl(G)\subset \text{int}(Cl(Q))\). Therefore, \(f(O)\subset \text{int}(Cl(Q))\).

**Characterization of faintly \(\delta\)-semicontinuity.**

**Definition 5:**

\(f\) : (\(X\), \(\tau\)) \(\rightarrow\) (\(Y\), \(\sigma\)) is A faintly \(\delta\)-\(S\)-continuous if \(\forall \chi\in X\) and all \(\delta\)-open subset \(Q\) of \(Y\) that contains \(f(\chi)\), \(\exists\) a \(\delta\)-semi open subset \(O\) of \(X\) that contains \(\chi\) s.t. \(f(O)\subseteq Q\).

**Theorem 7:**

Given \(f:(X, \tau) \rightarrow (Y, \sigma)\), then we can establish the equivalence of the following:

i. The map \(f\) is faintly \(\delta\)-\(S\)-continuous.

ii. The pre-image \(f^{-1}(Q)\) is a \(\delta\)-semi open subset of \(X\) for all \(r\)-open set \(Q\) of \(Y\).

iii. The pre-image \(f^{-1}(F)\) is a \(\delta\)-semi closed subset of \(X\) for all \(r\)-closed subset \(F\) of \(Y\).

iv. \(f(sCl(\delta)(E)) \subset Cl(f(E))\) \(\forall E \subset \chi\).

v. \(sCl(\delta)(f^{-1}(B)) \subset f^{-1}(Cl(\delta)(B))\) \(\forall B \subset Y\).

vi. The pre-image \(f^{-1}(F)\) is a \(\delta\)-semi closed subset in \(X\) \(\forall\) \(\delta\)-closed subset \(F\) of \(Y\).

vii. The pre-image \(f^{-1}(Q)\) is a \(\delta\)-semi open subset in \(X\) \(\forall\) \(\delta\)-open subset \(Q\) of \(Y\).

**Proof:**

(i\(\Rightarrow\)ii) for an \(r\)-open subset \(Q\) of \(Y\) suppose that \(\chi\in f^{-1}(Q)\), then \(Q = \text{int}(Cl(Q))\) and \(f(\chi)\in Q\), then \(Q\) is a \(\delta\)-open set of \(Y\) and by application of part (i) and the definition 5, \(\exists\) a \(\delta\)-semi open subset set \(O_X\) of \(X\) s.t. \(\chi\in O_X\) and \(f(O_X)\subset Q\). Therefore, \(\chi\in O_X\subset f^{-1}(f(O_X))\subset f^{-1}(Q)\) and \(\exists\) a \(\delta\)-open set \(W_X\) s.t. \(W_X\subset O_X\subset Cl(W_X)\), since \(O_X\) is \(\delta\)-semi open. Now, suppose that \(W=\cup_{\delta(\text{E}f)(Q)} W_X\). As \(\cup_{\delta(\text{E}f)(Q)} Cl(W_X)\subset Cl(W),\) then \(O=\cup_{\delta(\text{E}f)(Q)} (O_X) = f^{-1}(Q)\) and \(\delta\)-openness is established.

(ii\(\Rightarrow\)iii) Suppose that \(F\) is an \(r\)-closed subset \(F\) of \(Y\). \(\Rightarrow\) \(\forall\) \(r\)-open subset \(Y\). By part (ii), \(f^{-1}(Y)\) is a \(\delta\)-open subset of \(X\). Since \(f^{-1}(Y\setminus F) = f^{-1}(Y)\setminus f^{-1}(F)\), hence \(f^{-1}(F)\) is a \(\delta\)-semi closed subset of \(X\).

(iii\(\Rightarrow\)iv) Suppose that \(\chi\in sCl(\delta)(E)\) and suppose that \(f(\chi)\notin Cl(\delta)(f(E))\). Then, \(\exists\) an open set \(Q_0\) s.t. \(f(\chi)\in Q_0\) and \(f(Q_0)\subset \text{int}(Cl(Q_0))\) = \(\emptyset\). Then, take the \(r\)-open set \(W_0 = \text{int}(Cl(Q_0))\). Hence, \(f(E)\subset Y\{0\}.\) By part (iii), it got \(f^{-1}(Y\{0\})\) is \(\delta\)-semi closed and \(E\subset f^{-1}(Y\{0\})\). Thus, by the definition of \(sCl(\delta)(E)\), it got \(\chi\in f^{-1}(Y\{0\})\) a contradiction with \(f(\chi)\in Q_0\subset \text{int}(Cl(Q_0)) = W_0\).

(iv\(\Rightarrow\)v) Suppose that \(E = f^{-1}(B)\subset \chi\). Then, by part (iv) take \(f(sCl(\delta)(E))\subset Cl(\delta)(f(E))\). Since \(Cl(\delta)(f(E))\subset Cl(\delta)(B)\), it follows that \(sCl(\delta)(E)\subset f^{-1}(Cl(\delta)(B))\). (v\(\Rightarrow\)vi) Suppose \(F\) is a \(\delta\)-closed subset in \(Y\). This means \(F\subset Cl(F)\subset Cl(\delta)(F)\). Taking \(B\) = \(F\) in part (v), and it got \(sCl(\delta)(f^{-1}(F))\subset f^{-1}(F)\). As \(f^{-1}(F)\subset Cl(\delta)(f^{-1}(F))\) and \(sCl(\delta)(f^{-1}(F))\) is \(\delta\)-semi closed which concludes \(f^{-1}(F)\) is a \(\delta\)-semi closed subset of \(X\).

(vi\(\Rightarrow\)vii) Let \(Q\) be an \(\delta\)-open subset of \(Y\). Taking \(F = Y\{Q\} \subset\) part (vi) it got \(f^{-1}(Y\{Q\})\) = \(f^{-1}(Y\{Q\})\) is \(\delta\)-semi closed subset of \(X\). Thus, \(f^{-1}(Q)\) is \(\delta\)-open subset of \(X\).

(vii\(\Rightarrow\)i) Assume that \(\chi\in X\) and suppose that \(Q\) is \(\delta\)-open subset of \(Y\), \(f(\chi)\in Q\). By part (vii), \(f^{-1}(Q)\) is \(\delta\)-semi open subset of \(X\). Then, taking \(O = f^{-1}(Q)\), it got \(\chi\in O\) and \(f(O)\subset f(f^{-1}(Q))\subset Q\). Therefore, faint \(\delta\)-\(S\)-continuity of \(f\) is established.

**Theorem 8:** For any function between two spaces \(f: X \rightarrow Y\). If the graph function \(g\) is faintly \(\delta\)-\(S\)-continuous, then so is \(f\).

**Proof:** Let \(\chi\in X\) and assume that \(Q\) is \(\delta\)-\(r\)-open set that contains \(f(\chi)\). Then \(X\{Q\}\) is \(\delta\)-\(r\)-open subset of \(X\times Y\). [Theorem 5 in\(^1\)], it further contains \(g(\chi) = (\chi, f(\chi))\). Therefore, \(\exists\) \(O\) \(\delta(\text{S}(x))\) containing \(\chi\) \(s.t\) \(g(\chi)\subset X\times Q\), which implies \(f(\chi)\subset Q\), and faint \(\delta\)-\(S\)-continuity of \(f\) is established.

**Theorem 9:** If \(f: X \rightarrow Y\) is faintly \(\delta\)-\(S\)-continuous with \(Y\) almost regular. Then for all \(\chi\in X\) and \(\delta\)-open subset \(Q\) of \(Y\), \(f(\chi)\in Q\), \(\exists\) a \(\delta\)-open subset \(O\) in \(X\), \(\chi\in O\) s.t \(f(O)\subset \text{int}(Cl(Q))\).

**Proof.** If \(\chi\in X\) and \(Q\) is a \(\delta\)-open subset in \(Y\) with \(f(\chi)\in Q\), but \(Y\) is almost regular. \(\exists\) \(r\)-open subset \(G\) in \(Y\) s.t \(f(\chi)\in G\subset Cl(G)\subset \text{int}(Cl(Q))\). Since \(f\) is faintly \(\delta\)-\(S\)-continuous, with \(G\) is \(r\)-open, then \(G\) is \(\delta\)-open. It follows \(\exists\) a \(\delta\)-\(S\)-open subset \(O\) of \(X\), with \(\chi\in O\) s.t \(f(O)\subset Cl(G)\subset \text{int}(Cl(Q))\).

**Remark 1:**

Clearly, any union \(\delta\)-open sets in \((X, \tau)\) is \(\delta\)-open. However, as can be seen in the example.
below, the result for intersection is generally false.

**Example 1:**
Suppose $\mathbb{R}^2$ with the usual topology. Suppose that $E$ be the set defined by $E= \{(X, Y) \in \mathbb{R}^2: X^2+Y^2 < 1\} \cup \{(\cos(\alpha), \sin(\alpha)) : 0 < \alpha \leq \pi/2 \}$. Then suppose that $G, H \in E$ further that $G \in \delta SO(H)$ and $H \in \delta O(X)$, then $G \in \delta SO(X)$.

**Theorem 11:**
Suppose that $f: (X, \tau) \to (Y, \sigma)$ is a mapping with $\{Q_i : i \in I\}$ an $\delta$-open cover of $X$. If the restriction $f|Q_i: (Q_i, \tau Q_i) \to (Y, \sigma)$ is faintly $\delta$-S-continuous $\forall i \in I$, then $f$ is faintly $\delta$-S-continuous.

**Proof:** If $O$ is an $\delta$-open set in $(Y, \sigma)$ (By Lemma 1). Therefore, $f^{-1}(O) = \bigcap f^{-1}(O) \subseteq \bigcap f^{-1}(O)$, $i \in I$. By Lemma 2, for all $i \in I$, $f|Q_i^{-1}(O)$ is $\delta$-semi open in $X$ and as $f^{-1}(O)$ is $\delta$-semi open in $X$. Therefore, $f$ is faintly $\delta$-S-continuous.

**Definition 6:**
A mapping $f: X \to Y$ is An almost $\delta$-semi open if $f(Q) \subseteq \text{int}(\text{Cl}(f(Q))) \forall \delta$-semi open subset $Q$ of $X$.

**Theorem 12:**
Given a mapping $f: X \to Y$ that is faintly $\delta$-S-continuous and almost $\delta$-semi open, then for all $\chi \in X$ and all $\delta$-open set $O \subseteq Y$, $\exists$ a $\delta$-semi open set $Q \subseteq \delta SO(X)$ s.t $f(Q) \subseteq \text{int}(\text{Cl}(O))$.

**Proof:** Let $\chi \in X$ and suppose that $O$ is an $\delta$-open subset of $Y$ s.t $f(\chi) \in O$. By faint $\delta$-S-continuity of $f$, there is $Q \subseteq \delta SO(X)$ s.t $f(Q) \subseteq O$. But $f$ is almost $\delta$-open, which implies that $f(Q) \subseteq \text{int}(\text{Cl}(f(Q))) \subseteq \text{int}(\text{Cl}(O))$, then $f(Q) \subseteq \text{int}(\text{Cl}(O))$.

**Note:** Many authors defined and introduced a generalization form of semiopen sets and semi closed sets have many applications see for example [16].

**Conclusion**
In this work, several results on faintly $\theta$ S-continuous and faintly $\delta$-S-continuous were obtained. Several properties of these kinds of faint continuity were considered. Also, the relations between the graph of faintly $\theta$ S-continuous and faintly $\delta$-S-continuous functions were obtained. Furthermore, the relation between these types of functions was considered.

**Authors’ Declaration**
- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.
Authors’ Contribution Statement

Sh. H. A. gives some results on faintly θ S-continuous and faintly δs-continuous function. J. H. H. introduces the mapping named “faintly δ-S-continuous” and give some results on it. A. M. Z. give several properties of faintly θ S-continuous and faintly δ-S-continuous.

References
حوَلِ الدوال المستمرة الضعيفة من النمط و

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الخلاصة
الدوال المستمرة بشكل ضعيف (FC) والمعروفة باسم الدوال شبه المستمرة بشكل ضعيف من النوع S و S، تعتبر من الخصائص والمميزات للدوال شبه المستمرة بشكل ضعيف من النوع S، تم دراستها والتحقق منها بوساطة المجاميع المفتوحة من النوع S. و S، والدوال شبه المستمرة بشكل ضعيف من النوع S، تم الحصول عليها. أضافة إلى ذلك، العلاقة بين الدوال المستمرة بشكل ضعيف من النوع S، و S، والدوال شبه المستمرة بشكل ضعيف من النوع S، و S، تم دراستها والتحقق منها أيضاً، كما أن الدالة شبه مستمرة بشكل ضعيف من النوع S، هي دالة شبه مستمرة ضعيفة S، لذا، تم العكس للنتيجة المادرة الذكر يتحقق عندما يكون المجال المقابل للدالة من النوع المنتظم تقريباً.

الكلمات المفتاحية: دالة من النوع S، صيغة شبه مستمرة، دالة من النوع S، مجموعة من النوع S، شبه متؤولة، دالة من النوع S، صيغة شبه مستمرة، مجموعة من النوع S، شبه متؤولة، إستمرارية ضعيفة.